Fundamental Groups

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Introduction

Intuitively, the fundamental group, $\pi(X)$ of a topological space X is a group of homotopy classes of closed loops on X that share a common base point. Though the fundamental group of a topological space is independent (up to isomorphism) of the base point chosen, the base point is necessary to define a binary operation for $\pi(X)$.

Formally, we define a closed loop on a topological space X with base point x_0 as a continuous mapping $f : [0, 1] \mapsto X$ such that $f(0) = x_0$ and $f(1) = x_0$.

We will consider two loops f and g to be equivalent if there exists a homotopy (that is, a continuous function) $H : [0, 1] \times [0, 1] \mapsto X$ such that if $x \in [0, 1]$ then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. This homotopy can be thought of as a continuous deformation from the loop f to the loop g that takes place in finite time and does not leave X.

We define the product of two loops $f(t)$ and $g(t)$ thus:

$$(f \cdot g)(t) = \left\{ \begin{array}{cc} f(2t), & 0 \leq t \leq 1/2 \\ g(2t-1), & 1/2 \leq t \leq 1 \end{array} \right\}$$

This describes traversing the second loop after traversing the first. The time taken to traverse each loop is cut in half in order to produce a loop that meets our membership criteria for $\pi(X)$.

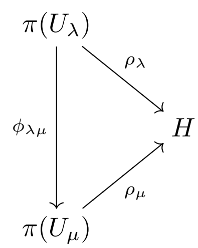
We define the inverse of a loop $f(t)$ as the same loop, traversed backwards. That is, $f^{-1}(t) = f(1-t)$. Traversing a loop forwards then backwards is equivalent to traversing the trivial loop, e, which is homotopic to the base point x_0.

The Seifert and Van Kampen Theorem

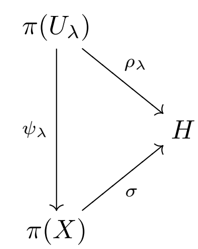
Conceptually, the Seifert and Van Kampen Theorem describes the construction of fundamental groups of complicated spaces from those of simpler spaces. To find the fundamental group of a topological space X using the Seifert and Van Kampen theorem, one covers X with a set of open, arcwise-connected subsets that is closed under finite intersection. One then takes the free product of the fundamental groups of the subsets in the covering to form $\pi(X)$. The proofs that follow can be found in W. S. Massey's *Algebraic Topology: An Introduction*, though I have rewritten them to include additional explanations.

Let X be an arcwise-connected topological space and $x_0 \in X$. Let $\{U_{lambda} : \lambda \in \Lambda\}$ be a covering of X by arcwise-connected open sets such that for all $\lambda \in \Lambda, x_0 \in U_\lambda$. Let this covering be closed under finite intersection. That is, for any two indices $\lambda_1, \lambda_2 \in \Lambda$ there exists an index $\lambda \in \Lambda$ such that $U_{\lambda_1} \cap U_{\lambda_2} = U_\lambda$.

Theorem (Seifert and Van Kampen): *Under the above hypotheses, the group $\pi(X)$ satisfies the following universal mapping condition: Let H be any group and let $\rho_\lambda : \pi(U_\lambda) \mapsto H$ be any collection of homomorphisms defined for all $\lambda \in \Lambda$ such* *that if $U_\lambda \subset U_\mu$, the following diagram is commutative:*



Then, there exists a unique homomorphism $\sigma : \pi(X) \mapsto H$ such that for any $\lambda \in \Lambda$ the following diagram is commutative:



Moreover, this universal mapping condition characterizes $\pi(X)$ up to a unique isomorphism.

The fact that this mapping property characterizes $\pi(X)$ follows from the first isomorphism theorem. By forming the quotient $H/\ker(\sigma)$, we produce a group that is isomorphic to $\pi(X)$.

To prove this theorem, we begin with a lemma.

Lemma:*The group $\pi(X)$is generated by the union of the images $\psi_\lambda[\pi(U_\lambda)], \lambda \in \Lambda$*

Proof: Let $\alpha \in \pi(X)$ and let f be a lift of α (that is, a closed loop on X in the homotopy class α). Choose an integer n sufficiently large so that $1/n$ is less than the Lebesgue number of the open covering $\{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ of the compact metric space I. Subdivide the interval I into the closed subintervals $J_i = [i/n, i+1/n], 0 \leq i \leq n-1$. Because our choice of n guarantees that each J_i lies entirely in at least one subset $f^{-1}(U_\lambda)$ of the covering, we may choose an index $\lambda_i \in \Lambda$ such that $f(J_i) \subset U_{\lambda_i}$. Choose a path g_i in $U_{\lambda_{i-1}} \cap U_{\lambda_i}$ joining the base point x_0 to the point $f(i/n)$, $1 \leq i \leq n-1$. We proceed to let $f_i : I \mapsto X$ denote the path represented by the composite function

$$I \xrightarrow{h_i} J_i \xrightarrow{f|J_i} X$$

where h_i is the unique orientation-preserving linear homomorphism. Intuitively, f_i is the image of one interval J_i through f. The path f_i begins at $f(i/n)$ and ends at $f(i+1/n)$. We can form closed paths of the following form: $f_0 \cdot g_1^{-1}, g_1 \cdot f_1 \cdot g_2^{-1}, g_2 \cdot f_2 \cdot g_3^{-1}, ..., g_{n-1} \cdot f_{n-1}$. Each g_i begins at x_0 and ends at $f(i/n)$. f_1 begins there and ends at $f(i+1/n)$. g_{i+1}^{-1} begins there and ends back at x_0. The resulting paths are each contained in a single open set U_λ, and their product in the order given is equivalent to f. Hence we can write

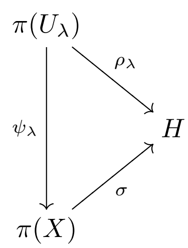
$$\alpha = \alpha_0 \cdot \alpha_1 \cdot \alpha_2 \cdot ... \cdot \alpha_{n-1},$$

where

$$\alpha_i \in \psi_{\lambda_i}[\pi(U_{\lambda_i})], 0 \leq i \leq n-1.$$

Since we have shown that any $\alpha \in \pi(X)$ can be expressed as the product of some $\{\alpha_i\}$, images of elements of $\pi(U_\lambda)$, we have shown that the group $\pi(X)$is indeed generated by the union of the images $\psi_\lambda[\pi(U_\lambda)], \lambda \in \Lambda$

Proof (of Seifert and Van Kampen): Let H be any group and let $\{\rho_\lambda : \pi(U_\lambda) \mapsto H, \lambda \in \Lambda\}$ be a set of homomorphisms satisfying the hypotheses of the theorem. We must demonstrate the existence of a unique homomorphism $\sigma : \pi(X) \mapsto H$ such that the following diagram is commutative for any $\lambda \in \Lambda$:



Following the lemma just proved, the homomorphism σ, if it exists, must be defined as follows. Let $\alpha \in \pi(X)$. Using the lemma,

$$\alpha = \psi_{\lambda_1}(\alpha_1) \cdot \psi_{\lambda_2}(\alpha_2) \cdot ... \cdot \psi_{\lambda_n}(\alpha_n)$$

where $\alpha_i \in \pi(X), i = 1, 2, ..., n$. Tracing the diagram and using the fact that σ, if it exists, is supposed to be a homomorphism, we must have:

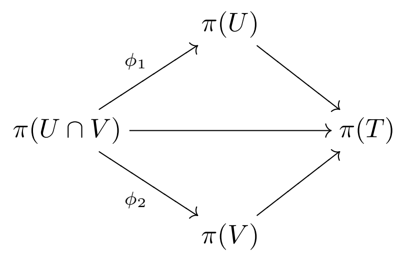
$$\sigma(\alpha) = \rho_{\lambda_1}(\alpha_1) \cdot \rho_{\lambda_2}(\alpha_2) \cdot ... \cdot \rho_{\lambda_n}(\alpha_n)$$

.

The fact that our diagram must commute gives us the uniqueness of σ. What we do not yet have is certainty that σ is independent of the representation chosen for α. For the purposes of this exposition, we will assume this fact. Defining σ as above allows us to produce σ and thereby prove its existence.

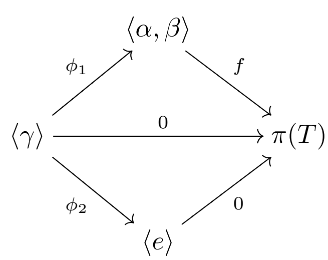
Applications

Torus: To see the Seifert and Van Kampen Theorem in action, we shall construct the fundamental group of the torus T. Let us choose the covering consisting of the punctured torus, soit U, an open disk that covers the puncture, soit V, and their intersection (a punctured disk). The Seifert and Van Kampen Theorem gives us the following diagram where ϕ_i denotes the inclusion map:



We know that $\pi(V)$ is trivial since V is simply connected. Imagining the punctured torus as its fundamental polygon minus a point makes it clear that the punctured torus retracts to just the edge of that square with edges identified, and thus to a figure eight. Since U retracts to a figure eight, we know that $\pi(V)$ is the free group on two generators, α and β (these generators being the equivalences classes of single loops around either leaf of the figure eight). $U \cap V$ is a punctured disk, and therefore retracts to a circle. Thus, $\pi(U \cap V)$ is infinite cyclic on one generator γ. Visualizing the punctured torus once again as its fundamental polygon, any lift of $\phi_1(\gamma)$ is a loop around the puncture. Deforming this loop to the edge of the square, it can be seen that $\phi_1(\gamma) = \alpha\beta\alpha^{-1}\beta^{-1}$.

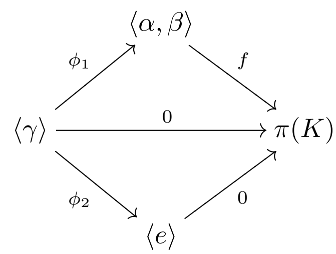
Considering that $\pi(V)$ is trivial, it must map to the trivial element of $\pi(T)$. This allows us to further label our diagram.



Because this diagram commutes, we know that $\pi(U) = \langle \alpha, \beta \rangle$ must map into $\pi(T)$ such that $\phi_1(\gamma) = \langle \alpha\beta\alpha^{-1}\beta^{-1} \rangle$ is in the kernel of that map. If the kernel were any bigger, f would not surject, $\pi(T)$ would have more generators than our covering lends it, and our lemma would be contradicted. Therefore, we know that $\pi(T)$ is the free group on two generators, mod its commutator subgroup. This tells us that $\pi(T)$ is the free abelian group on two generators and is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Klein Bottle: We proceed with the Klein bottle much like we did with the torus. Let us construct our open covering of the Klein bottle K consisting of K minus a point (call it U), an open patch that includes the missing point (call it V), and the intersection of the two.

Consider U as the fundamental polygon of K minus a point. Retracting this to the edges of the square and identifying edges appropriately, we obtain a figure eight again and learn that the fundamental group of U is the free group on two generators α and β. Once again, the fundamental group of V is trivial. As before, the fundamental group of $U \cap V$ is infinite cyclic with generator γ, whose lifts are single loops around the puncture in $U \cap V$. Considering the image of one of these loops on the fundamental polygon of K, we can deform it to the edge and see that it is equivalent to $\alpha\beta\alpha\beta^{-1}$. We label our diagram as before:



And find, by the reasoning previously presented, that $\ker(f) = \langle\alpha\beta\alpha\beta^{-1}\rangle$. This makes $\pi(K)$ isomorphic to $\mathbb{Z} \times \mathbb{Z}/\langle\alpha\beta\alpha\beta^{-1}\rangle$.

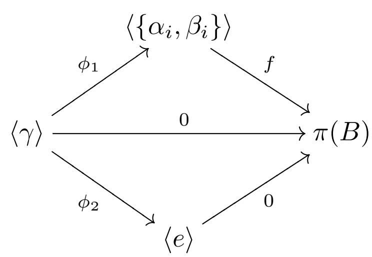
It's interesting that even though the fundamental groups of the open sets in covering of the torus are the same as those of the Klein bottle, the way that they map into each other is different. Because of this, the torus and the Klein bottle do not have the same fundamental group.

The Connected Sum of n Klein Bottles: As before, we consider the fundamental polygon of our space. In this case, we have a $4n-$gon whose edges are (in order from an arbitrary starting point)

$$\alpha_1, \beta_1, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_2, \beta_2, ..., \alpha_n, \beta_n, \alpha_n, \beta_n$$

. Let these edges be oriented such that, in each set of four, the first appearance of α_i faces clockwise, the second faces counterclockwise, and both appearances of β_i face clockwise.

We proceed to cover our space in the usual way. Let B be the connected sum of n Klein bottles. Let U be B minus one interior point of the polygon. Let V be an open patch covering the puncture. Retracting our punctured polygon to its edge, we find that the fundamental group of U is the free group on $2n$ generators, $\{\alpha_i, \beta_i : i \in \{1, 2, ...n\}\}$. The fundamental group of V is trivial, and that of $U \cap V$ is free on one generator γ. We construct our usual diagram thus:



This time,

$$\phi_1(\gamma) = \prod_{i=1}^{n} \alpha_i \beta_i \alpha_i^{-1} \beta_i$$

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As before, the subgroup generated by $\phi_1(\gamma)$ is the kernel of f. This makes $\pi(B)$ isomorphic to

$$\mathbb{Z}^n / \langle \prod_{i=1}^{n} \alpha_i \beta_i \alpha_i^{-1} \beta_i \rangle$$

Commentary

Fundamental groups are useful tools that can be used to help describe a topological space. A space is simply connected if and only if its fundamental group is trivial. In some sense, the fundamental group represents the degree to which a space fails to be simply connected.

Fundamental groups are also useful in determining whether two spaces are not homeomorphic. Homeomorphic spaces have the same fundamental group. It should be noted, however, that the converse is not true.

Determining the fundamental group of a topological space using the Seifert and Van Kampen Theorem relies not only on what the fundamental groups of the smaller spaces are, but how they map into each other. This is a prime example of how a mapping-theoretic approach can yield tremendous insight into the structure of something. On the face of it, one might think that the torus and the Klein bottle would have the same fundamental group, given that you can cover them in such similar ways. It is when one considers the mappings between pieces of the covering that one discovers the difference.