

THE p -ADIC INTEGERS CLASSICALLY AND AS A LIMIT

1. THE p -ADIC INTEGERS CLASSICALLY

- Define p -adic valuation on \mathbf{Z} :

$$\begin{aligned}\nu_p(0) &= +\infty, \\ \nu_p(p^e m) &= e, \quad e \in \mathbf{Z}, \quad m \in \mathbf{Z} - p\mathbf{Z}.\end{aligned}$$

Then

$$\begin{aligned}\nu_p(xy) &= \nu_p(x) + \nu_p(y), \\ \nu_p(x + y) &\geq \min\{\nu_p(x), \nu_p(y)\}, \quad \text{equality if } \nu_p(x) \neq \nu_p(y).\end{aligned}$$

Next define p -adic absolute value on \mathbf{Z} :

$$|x|_p = p^{-\nu_p(x)}, \quad x \in \mathbf{Z}.$$

Then for all $x, y \in \mathbf{Z}$,

$$\begin{aligned}|xy|_p &= |x|_p \cdot |y|_p, \\ |x + y|_p &\leq \max\{|x|_p, |y|_p\}, \quad \text{equality if } |x|_p \neq |y|_p.\end{aligned}$$

Next define p -adic distance on \mathbf{Z} ,

$$d_p(x, y) = |x - y|_p.$$

Thus for all $x, y, z \in \mathbf{Z}$,

$$\begin{aligned}d(x, y) &\geq 0 \quad \text{with equality if and only if } x = y, \\ d(x, y) &= d(y, x), \\ d(x, y) &\leq d(x, z) + d(y, z).\end{aligned}$$

Complete the metric space $(\mathbf{Z}, |\cdot|_p)$ to get the p -adic integers. Why is \mathbf{Z}_p compact? (Heine–Borel *not* applicable.)

2. CONTINUITY

- ε - δ continuity of $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$.
- Open set rephrase. Open sets of \mathbf{R}^n generated by open boxes.
- ε - δ continuity of $f : A \rightarrow \mathbf{R}^m$. Subspace topology of A .

3. TOPOLOGY

- Topological space (X, \mathcal{T}) :
 - $X \in \mathcal{T}, \emptyset \in \mathcal{T}$.
 - $\bigcup_i \mathcal{O}_i \in \mathcal{T}$.
 - $\bigcap_{i=1}^n \mathcal{O}_i \in \mathcal{T}$.
- Topological continuity. No (direct) reference to a metric.

4. PRODUCTS

- Which sets should generate the open subsets of $\prod_i X_i$?
- Explanation from the set-theoretic viewpoint?
- Explanation from the mapping-property viewpoint.
- Tychonoff Theorem. General.

5. LIMITS

- Limit topology is natural: subspace topology of product topology.
- $\mathbf{Z}_p = \lim_e \mathbf{Z}/p^e \mathbf{Z}$: Compactness now naturally shown since

$$\mathbf{Z}_p = \bigcap_{e \geq 1} \text{graph}(\mathbf{Z}/p^e \mathbf{Z} \longrightarrow \mathbf{Z}/p^{e-1} \mathbf{Z}).$$

Closed subspace of compact space is again compact.