MATHEMATICS 332: ALGEBRA — ASSIGNMENT 3

Reading: Gallian, chapters 4, 7, 9

Problems:

1. Let G be a group, and let H be a subgroup of index 2. Show that H is a normal subgroup of G.

2. The *center* of a group G is

$$Z(G) = \{c \in G : cg = gc \quad \text{for all } g \in G\}.$$

(a) Show that $Z(G)$ is a normal subgroup of G.
(b) Determine, with justification, the center of $\mathrm{GL}_n(\mathbb{R})$.

3. Consider a subgroup of $\mathrm{GL}_2(\mathbb{R})$,

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \right\}.$$

Consider also two subsets of G,

$$H_1 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in G : ad = 1 \right\}, \qquad H_2 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in G : d = 1 \right\}.$$

Prove that H_1 and H_2 are normal subgroups of G by finding homomorphisms f_1 and f_2 having H_1 and H_2 as their kernels. (Note: The homomorphisms need not map G back to G.)

4. Let p be an odd prime, and let H be a cyclic group of order p. Consider a set

$$G = \{(a, x) : a \in \pm 1,\ x \in H\}$$

and a composition law

$$(a, x)(b, y) = (ab, xy^a).$$

(a) Show that the composition law makes G a group of order $2p$.
(b) How many subgroups does G have?

5. Let $n \geq 1$ be an integer. Let $G = \mathrm{SO}_{n+1}(\mathbb{R})$, and consider a subgroup of G,

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} : b \in \mathrm{SO}_n(\mathbb{R}) \right\}.$$

Consider the unit n-sphere in $\mathbb{R}^{n+1}$ and one of its points,

$$S = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_{n+1} \end{bmatrix} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}, \qquad v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Explain why there is a bijection

$$G/H \approx S.$$