CANONICAL FORMS IN LINEAR ALGEBRA

Let k be a field, let V be a finite-dimensional vector space over k, and let

$$T : V \longrightarrow V$$

be an endomorphism. Linear algebra teaches us, laboriously, that T has a *rational canonical form* and (if k is algebraically closed) a *Jordan canonical form*. This writeup shows that both forms follow quickly and naturally from the structure theorem for modules over a PID.

1. The Module Associated to T

Since k is a field, the polynomial ring $k[X]$ is a PID. Give V the structure of a $k[X]$-module by defining

$$f(X) \cdot v = f(T)v, \quad f(X) \in k[X].$$

Especially, X acts as T. The structure theorem for modules over a PID says that

$$V \approx \frac{k[X]}{\langle f_1(X)\rangle} \oplus \cdots \oplus \frac{k[X]}{\langle f_m(X)\rangle}, \qquad f_1(X) \mid \cdots \mid f_m(X).$$

That is, $V = \bigoplus_i V_i$, where for each i there is an isomorphism

$$k[X]/\langle f_i(X)\rangle \longrightarrow V_i$$

that intertwines the action of $k[X]$ on itself and the action of $k[T]$ on V_i. Thus for each polynomial $g(X)$ with $\deg(g) < \deg(f_i)$ there exists a unique $v \in V_i$ so that the following square commutes,

$$\begin{array}{ccc} v & \overset{T}{\longmapsto} & Tv \\ \uparrow & & \uparrow \\ g(X) + \langle f_i(X)\rangle & \overset{X}{\longmapsto} & Xg(X) + \langle f_i(X)\rangle \end{array}$$

In particular, the unique vector $v_i \in V_i$ corresponding to $g(X) = 1$ in the diagram has the property

$$X^j + \langle f_i(X)\rangle \longmapsto T^j v_i \quad \text{for } j = 0, 1, 2, \ldots.$$

Let $d_i = \deg(f_i)$. Since $\{1, X, X^2, \ldots, X^{d_i-1}\}$ is a canonical basis of $k[X]/\langle f_i(X)\rangle$ (now writing representatives rather than cosets for tidiness), correspondingly

$$\{v_i, Tv_i, T^2v_i, \ldots, T^{d_i-1}v_i\} \quad \text{is a canonical basis of } V_i.$$

That is, courtesy of the modules over a PID structure theorem, T acts as multiplication by X in "polynomial coordinates," and so V_i has a canonical cyclic basis generated by a vector v_i and its images under iterates of T. This is the crux of everything to follow.

2. Rational Canonical Form

To obtain the rational canonical form of T, let each f_i have degree d_i. More specifically,

$$f_i(X) = a_{0,i} + a_{1,i}X + a_{2,i}X^2 + \cdots + a_{d_i-1,i}X^{d_i-1} + X^{d_i}.$$

A basis for the ith summand $k[X]/\langle f_i(X)\rangle$ is (again abbreviating cosets to their representatives)

$$\{1, X, X^2, \ldots, X^{d_i-1}\}.$$

As explained a moment ago, the corresponding basis of the summand V_i is T-*cyclic*,

$$\{v_i, Tv_i, T^2v_i, \ldots, T^{d_i-1}v_i\},$$

while, because $T^{d_i}v_i$ corresponds to X^{d_i} and $f_i(X) = 0$ in $k[X]/\langle f_i(X)\rangle$,

$$T^{d_i}v_i = -a_{0,i}v_i - a_{1,i}Tv_i - \cdots - a_{d_i-1,i}T^{d_i-1}v_i.$$

And so with respect to this basis, the restriction of T to V_i has matrix

$$A_i = \begin{bmatrix} 0 & & & & & -a_{0,i} \\ 1 & 0 & & & & -a_{1,i} \\ & 1 & 0 & & & -a_{2,i} \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & -a_{d_i-1,i} \end{bmatrix}.$$

Consequently the restriction of T to V_i has characteristic polynomial $f_i(X)$, as can be seen by expanding $\det(XI - A_i)$ by cofactors across the top row. Thus T has characteristic polynomial $\prod_i f_i(X)$ and minimal polynomial $f_m(X)$.

3. Jordan Canonical Form

To obtain the Jordan canonical form of T, now take k to be algebraically closed. The factorization of each ith elementary divisor polynomial,

$$f_i(X) = \prod_{j=1}^{\ell_i}(X - \lambda_{ij})^{e_{ij}},$$

gives a decomposition of the ith polynomial quotient ring,

$$\frac{k[X]}{\langle f_i(X)\rangle} \approx \frac{k[X]}{\langle (X-\lambda_{i1})^{e_{i1}}\rangle} \oplus \cdots \oplus \frac{k[X]}{\langle (X-\lambda_{i\ell_i})^{e_{i\ell_i}}\rangle},$$

and then a corresponding decomposition of the ith cyclic subspace, $V_i = \bigoplus_j V_{ij}$.

A basis for the (i,j)th summand $k[X]/\langle (X-\lambda_{ij})^{e_{ij}}\rangle$ is (abbreviating cosets)

$$\{1, X - \lambda_{ij}, (X-\lambda_{ij})^2, \ldots, (X-\lambda_{ij})^{e_{ij}-1}\}.$$

The corresponding basis of V_{ij} is $(T - \lambda_{ij})$-cyclic, taking the form

$$\{v_{ij}, (T-\lambda_{ij})v_{ij}, (T-\lambda_{ij})^2v_{ij}, \ldots, (T-\lambda_{ij})^{e_{ij}-1}v_{ij}\}.$$

Fix i and j, and then drop them from the notation. Denote the basis elements in the previous display $v_0, v_1, \ldots, v_{e-1}$, and let $v_e = 0$. Then we have in V_{ij},

$$(T - \lambda)v_k = v_{k+1}, \quad k = 0, \ldots, e-1,$$

or

$$Tv_k = \lambda v_k + v_{k+1}, \quad k = 0, \ldots, e-1,$$

The previous display encompasses the relation $Tv_{e-1} = \lambda v_{e-1}$ since $v_e = 0$. Thus with respect to the basis, the restriction of T to V_{ij} has matrix

$$J_{i,j} = \begin{bmatrix} \lambda & & & & & \\ 1 & \lambda & & & & \\ & 1 & \lambda & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \lambda & \\ & & & & 1 & \lambda \end{bmatrix}.$$

Now let i and j vary. For any $\lambda \in k$, the algebraic multiplicity of λ is the number of times that it repeats as a root of the characteristic polynomial $\prod_i f_i(X)$; the geometric multiplicity of λ (and the number of λ-blocks in the Jordan form of the endomorphism T) is the number of factors $f_i(X)$ of which it is a root; and the size of the ith λ-block is the multiplicity of λ as a root of $f_i(X)$.

4. An Example

Let $k = \mathbb{Q}$ and suppose that an endomorphism $T : V \longrightarrow V$ has characteristic polynomial

$$f(X) = (X^2 - 2)^4.$$

Then the 8-dimensional vector space V over $\mathbb{Q}$ takes one of the following possible forms as a $\mathbb{Q}[X]$-module:

$$V \approx \begin{cases} \mathbb{Q}[X]/\langle (X^2-2)^4 \rangle, \\ \mathbb{Q}[X]/\langle X^2-2 \rangle \oplus \mathbb{Q}[X]/\langle (X^2-2)^3 \rangle, \\ \mathbb{Q}[X]/\langle (X^2-2)^2 \rangle \oplus \mathbb{Q}[X]/\langle (X^2-2)^2 \rangle, \\ \mathbb{Q}[X]/\langle X^2-2 \rangle \oplus \mathbb{Q}[X]/\langle X^2-2 \rangle \oplus \mathbb{Q}[X]/\langle (X^2-2)^2 \rangle, \\ \mathbb{Q}[X]/\langle X^2-2 \rangle \oplus \mathbb{Q}[X]/\langle X^2-2 \rangle \oplus \mathbb{Q}[X]/\langle X^2-2 \rangle \oplus \mathbb{Q}[X]/\langle X^2-2 \rangle. \end{cases}$$

Since

$$\begin{aligned} (X^2-2)^4 &= X^8 - 8X^6 + 24X^4 - 32X^2 + 16, \\ (X^2-2)^3 &= X^6 - 6X^4 + 12X^2 - 8, \\ (X^2-2)^2 &= X^4 - 4X^2 + 4, \end{aligned}$$

the corresponding possible rational canonical forms of T are

$$\begin{array}{|cccccccr|} 0 & & & & & & & -16 \\ 1 & 0 & & & & & & 0 \\ & 1 & 0 & & & & & 32 \\ & & 1 & 0 & & & & 0 \\ & & & 1 & 0 & & & -24 \\ & & & & 1 & 0 & & 0 \\ & & & & & 1 & 0 & 8 \\ & & & & & & 1 & 0 \end{array} \quad \text{(having minimal polynomial } (X^2-2)^4\text{)},$$

and

$$\begin{array}{|cc|cccccc|}
\hline
0 & 2 & & & & & & \\
1 & 0 & & & & & & \\
\hline
 & & 0 & & & & & 8 \\
 & & 1 & 0 & & & & 0 \\
 & & & 1 & 0 & & & -12 \\
 & & & & 1 & 0 & & 0 \\
 & & & & & 1 & 0 & 6 \\
 & & & & & & 1 & 0 \\
\hline
\end{array} \quad \text{(having minimal polynomial } (X^2-2)^3\text{),}$$

and

$$\begin{array}{|cccc|cccc|}
\hline
0 & & & -4 & & & & \\
1 & 0 & & 0 & & & & \\
 & 1 & 0 & 4 & & & & \\
 & & 1 & 0 & & & & \\
\hline
 & & & & 0 & & & -4 \\
 & & & & 1 & 0 & & 0 \\
 & & & & & 1 & 0 & 4 \\
 & & & & & & 1 & 0 \\
\hline
\end{array} \quad \text{(having minimal polynomial } (X^2-2)^2\text{),}$$

and

$$\begin{array}{|cc|cc|cccc|}
\hline
0 & 2 & & & & & & \\
1 & 0 & & & & & & \\
\hline
 & & 0 & 2 & & & & \\
 & & 1 & 0 & & & & \\
\hline
 & & & & 0 & & & -4 \\
 & & & & 1 & 0 & & 0 \\
 & & & & & 1 & 0 & 4 \\
 & & & & & & 1 & 0 \\
\hline
\end{array} \quad \text{(having minimal polynomial } (X^2-2)^2\text{),}$$

and

$$\begin{array}{|cc|cc|cc|cc|}
\hline
0 & 2 & & & & & & \\
1 & 0 & & & & & & \\
\hline
 & & 0 & 2 & & & & \\
 & & 1 & 0 & & & & \\
\hline
 & & & & 0 & 2 & & \\
 & & & & 1 & 0 & & \\
\hline
 & & & & & & 0 & 2 \\
 & & & & & & 1 & 0 \\
\hline
\end{array} \quad \text{(having minimal polynomial } X^2-2\text{).}$$

For the Jordan form of T, consider a base field k that contains $\sqrt{2}$, and view T as an endomorphism of an 8-dimensional vector space $\widetilde{V}$ over k, so that its characteristic polynomial becomes

$$f(X) = (X-\sqrt{2})^4(X+\sqrt{2})^4.$$

Then the Jordan canonical form of the part of T associated to $\sqrt{2}$ is one of

$$\begin{array}{|cccc|} \hline \sqrt{2} & & & \\ 1 & \sqrt{2} & & \\ & 1 & \sqrt{2} & \\ & & 1 & \sqrt{2} \\ \hline \end{array} \quad \text{(having minimal polynomial } (X-\sqrt{2})^4\text{),}$$

or

$$\begin{array}{|ccc|c|} \hline \sqrt{2} & & & \\ 1 & \sqrt{2} & & \\ & 1 & \sqrt{2} & \\ \hline & & & \sqrt{2} \\ \hline \end{array} \quad \text{(having minimal polynomial } (X-\sqrt{2})^3\text{),}$$

or

$$\begin{array}{|cc|cc|} \hline \sqrt{2} & & & \\ 1 & \sqrt{2} & & \\ \hline & & \sqrt{2} & \\ & & 1 & \sqrt{2} \\ \hline \end{array} \quad \text{(having minimal polynomial } (X-\sqrt{2})^2\text{),}$$

or

$$\begin{array}{|cc|c|c|} \hline \sqrt{2} & & & \\ 1 & \sqrt{2} & & \\ \hline & & \sqrt{2} & \\ \hline & & & \sqrt{2} \\ \hline \end{array} \quad \text{(having minimal polynomial } (X-\sqrt{2})^2\text{),}$$

or

$$\begin{array}{|c|c|c|c|} \hline \sqrt{2} & & & \\ \hline & \sqrt{2} & & \\ \hline & & \sqrt{2} & \\ \hline & & & \sqrt{2} \\ \hline \end{array} \quad \text{(having minimal polynomial } X-\sqrt{2}\text{).}$$

Note that the second and third cases have different minimal polynomials and yet the same geometric multiplicity of the eigenvalue $\sqrt{2}$. Note that the third and fourth cases have the same minimal polynomial and yet different geometric multiplicities of the eigenvalue $\sqrt{2}$.

Similarly for the part of T associated to $-\sqrt{2}$. Thus altogether the transformation has 25 possible Jordan forms.

5. Another Example

Let $\dim_k V = 12$, and let the endomorphism $T : V \longrightarrow V$ have characteristic polynomial $(X-\lambda)^{12}$ where $\lambda \in k$. Suppose we know the following nullities (kernel dimensions):

$$\begin{aligned} \mathcal{N}(T-\lambda I) &= 5, \\ \mathcal{N}((T-\lambda I)^2) &= 9, \\ \mathcal{N}((T-\lambda I)^3) &= 11, \\ \mathcal{N}((T-\lambda I)^4) &= 12. \end{aligned}$$

Then the Jordan form of T has five blocks, four $(9-5)$ of which are at least 2-by-2 so that one is 1-by-1, two $(11-9)$ of which are at least 3-by-3 so that two are

2-by-2, and one $(12-11)$ of which is exactly 4-by-4 so that one is 3-by-3. Thus the Jordan form of T is

$$\begin{array}{|c|cc|cc|ccc|cccc|}
\hline
\lambda & & & & & & & & & & & \\
\hline
 & \lambda & & & & & & & & & & \\
 & 1 & \lambda & & & & & & & & & \\
\hline
 & & & \lambda & & & & & & & & \\
 & & & 1 & \lambda & & & & & & & \\
\hline
 & & & & & \lambda & & & & & & \\
 & & & & & 1 & \lambda & & & & & \\
 & & & & & & 1 & \lambda & & & & \\
\hline
 & & & & & & & & \lambda & & & \\
 & & & & & & & & 1 & \lambda & & \\
 & & & & & & & & & 1 & \lambda & \\
 & & & & & & & & & & 1 & \lambda \\
\hline
\end{array}\;.$$