Let $k$ be a field, let $V$ be a finite-dimensional vector space over $k$, and let $T : V \to V$ be an endomorphism. Linear algebra teaches us, laboriously, that $T$ has a rational canonical form and (if $k$ is algebraically closed) a Jordan canonical form. This writeup shows that both forms follow quickly and naturally from the structure theorem for modules over a PID.

1. The Module Associated to $T$

Since $k$ is a field, the polynomial ring $k[X]$ is a PID. Give $V$ the structure of a $k[X]$-module by defining $f(X) \cdot v = f(T)v, f(X) \in k[X]$. Especially, $X$ acts as $T$. The structure theorem for modules over a PID says that

$$V \cong \frac{k[X]}{(f_1(X))} \oplus \cdots \oplus \frac{k[X]}{(f_m(X))}, \quad f_1(X) \mid \cdots \mid f_m(X).$$

That is, $V = \bigoplus_i V_i$, where for each $i$ there is an isomorphism $k[X]/(f_i(X)) \to V_i$ that intertwines the action of $k[X]$ on itself and the action of $k[T]$ on $V_i$. Thus for each polynomial $g(X)$ with $\deg(g) < \deg(f_i)$ there exists a unique $v \in V_i$ so that the following square commutes,

$$\begin{array}{ccc}
v & \overset{T}{\longrightarrow} & Tv \\
\downarrow & & \downarrow \\
g(X) + (f_i(X)) & \overset{X}{\longrightarrow} & Xg(X) + (f_i(X))
\end{array}$$

In particular, the unique vector $v_i \in V_i$ corresponding to $g(X) = 1$ in the diagram has the property

$$X^j + (f_i(X)) \hookrightarrow T^j v_i \quad \text{for } j = 0, 1, 2, \ldots$$

Let $d_i = \deg(f_i)$. Since $\{1, X, X^2, \ldots, X^{d_i-1}\}$ is a canonical basis of $k[X]/(f_i(X))$ (now writing representatives rather than cosets for tidiness), correspondingly

$$\{v_i, Tv_i, T^2 v_i, \ldots, T^{d_i-1} v_i\} \quad \text{is a canonical basis of } V_i.$$

That is, courtesy of the modules over a PID structure theorem, $T$ acts as multiplication by $X$ in “polynomial coordinates,” and so $V_i$ has a canonical cyclic basis generated by a vector $v_i$ and its images under iterates of $T$. This is the crux of everything to follow.
To obtain the rational canonical form of $T$, let each $f_i$ have degree $d_i$. More specifically,
$$f_i(X) = a_{0,i} + a_{1,i} X + a_{2,i} X^2 + \cdots + a_{d_i-1,i} X^{d_i-1} + X^{d_i}.$$ A basis for the $i$th summand $k[X]/(f_i(X))$ is (again abbreviating cosets to their representatives)
$$\{ 1, X, X^2, \ldots, X^{d_i-1} \}.$$ As explained a moment ago, the corresponding basis of the summand $V_i$ is $T$-cyclic,
$$\{ v_i, Tv_i, T^2v_i, \ldots, T^{d_i-1}v_i \},$$ while, because $T^{d_i}v_i$ corresponds to $X^{d_i}$ and $f_i(X) = 0$ in $k[X]/(f_i(X))$,
$$T^{d_i}v_i = -a_{0,i}v_i - a_{1,i}Tv_i - \cdots - a_{d_i-1,i}T^{d_i-1}v_i.$$ And so with respect to this basis, the restriction of $T$ to $V_i$ has matrix
$$A_i = \begin{bmatrix}
0 & -a_{0,i} \\
1 & 0 & -a_{1,i} \\
& 1 & 0 & -a_{2,i} \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -a_{d_i-1,i}
\end{bmatrix}.$$ Consequently the restriction of $T$ to $V_i$ has characteristic polynomial $f_i(X)$, as can be seen by expanding $\det(\lambda I - A_i)$ by cofactors across the top row. Thus $T$ has characteristic polynomial $\prod_i f_i(X)$ and minimal polynomial $f_m(X)$.

3. JORDAN CANONICAL FORM

To obtain the Jordan canonical form of $T$, now take $k$ to be algebraically closed. The factorization of each $i$th elementary divisor polynomial,
$$f_i(X) = \prod_{i=1}^{d_i} (X - \lambda_{ij})^{e_{ij}},$$ gives a decomposition of the $i$th polynomial quotient ring,
$$\frac{k[X]}{\langle f_i(X) \rangle} \approx \frac{k[X]}{\langle (X - \lambda_{ij})^{e_{ij}} \rangle} \oplus \cdots \oplus \frac{k[X]}{\langle (X - \lambda_{ij})^{e_{ij}} \rangle},$$ and then a corresponding decomposition of the $i$th cyclic subspace, $V_i = \bigoplus_j V_{ij}$. A basis for the $(i,j)$th summand $k[X]/\langle (X - \lambda_{ij})^{e_{ij}} \rangle$ is (abbreviating cosets)
$$\{ 1, X - \lambda_{ij}, (X - \lambda_{ij})^2, \ldots, (X - \lambda_{ij})^{e_{ij}-1} \}.$$ The corresponding basis of $V_{ij}$ is $(T - \lambda_{ij})$-cyclic, taking the form
$$\{ v_{ij}, (T - \lambda_{ij})v_{ij}, (T - \lambda_{ij})^2v_{ij}, \ldots, (T - \lambda_{ij})^{e_{ij}-1}v_{ij} \}.$$ Fix $i$ and $j$, and then drop them from the notation. Denote the basis elements in the previous display $v_0, v_1, \ldots, v_{e-1}$, and let $v_e = 0$. Then we have in $V_{ij}$,
$$T^kv = v_{k+1}, \quad k = 0, \ldots, e-1,$$
or
$$Tv_k = \lambda v_k + v_{k+1}, \quad k = 0, \ldots, e-1.$$
The previous display encompasses the relation $Tv_{e-1} = \lambda v_{e-1}$ since $v_e = 0$. Thus with respect to the basis, the restriction of $T$ to $V_{ij}$ has matrix

$$J_{i,j} = \begin{bmatrix}
    \lambda & 0 & 0 & \cdots & 0 \\
    1 & \lambda & 0 & \cdots & 0 \\
    0 & 1 & \lambda & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 1 & \lambda
\end{bmatrix}. $$

Now let $i$ and $j$ vary. For any $\lambda \in k$, the algebraic multiplicity of $\lambda$ is the number of times that it repeats as a root of the characteristic polynomial $\prod_i f_i(X)$; the geometric multiplicity of $\lambda$ (and the number of $\lambda$-blocks in the Jordan form of the endomorphism $T$) is the number of factors $f_i(X)$ of which it is a root; and the size of the $i$th $\lambda$-block is the multiplicity of $\lambda$ as a root of $f_i(X)$.

4. An Example

Let $k = \mathbb{Q}$ and suppose that an endomorphism $T : V \rightarrow V$ has characteristic polynomial

$$f(X) = (X^2 - 2)^4.$$ 

Then the 8-dimensional vector space $V$ over $\mathbb{Q}$ takes one of the following possible forms as a $\mathbb{Q}[X]$-module:

$$V \cong \begin{cases}
\mathbb{Q}[X]/((X^2 - 2)^4), \\
\mathbb{Q}[X]/(X^2 - 2) \oplus \mathbb{Q}[X]/((X^2 - 2)^3), \\
\mathbb{Q}[X]/((X^2 - 2)^2) \oplus \mathbb{Q}[X]/((X^2 - 2)^2), \\
\mathbb{Q}[X]/(X^2 - 2) \oplus \mathbb{Q}[X]/(X^2 - 2) \oplus \mathbb{Q}[X]/((X^2 - 2)^2), \\
\mathbb{Q}[X]/(X^2 - 2) \oplus \mathbb{Q}[X]/(X^2 - 2) \oplus \mathbb{Q}[X]/(X^2 - 2) \oplus \mathbb{Q}[X]/(X^2 - 2).
\end{cases}$$

Since

$$(X^2 - 2)^4 = X^8 - 8X^6 + 24X^4 - 32X^2 + 16,$$

$$(X^2 - 2)^3 = X^6 - 6X^4 + 12X^2 - 8,$$

$$(X^2 - 2)^2 = X^4 - 4X^2 + 4,$$

the corresponding possible rational canonical forms of $T$ are

$$
\begin{array}{cccccc}
0 & -16 & & & & \\
1 & 0 & & & & \\
1 & 0 & 32 & & & \\
1 & 0 & 0 & & & \\
1 & 0 & & -24 & & \\
1 & 0 & 0 & & & \\
1 & 0 & 0 & 8 & & \\
1 & 0 & & & & \\
\end{array}
$$

(having minimal polynomial $(X^2 - 2)^4$).
and

\[
\begin{pmatrix}
0 & 2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 8 \\
1 & 0 \quad 0 \\
1 & 0 \quad -12 \\
1 & 0 \quad 0 \\
1 & 0 \quad 6 \\
1 & 0
\end{pmatrix}
\text{(having minimal polynomial } (X^2 - 2)^3),
\]

and

\[
\begin{pmatrix}
0 & 0 & -4 \\
1 & 0 & 0 \\
1 & 0 & 4 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 8 \\
1 & 0 \quad 0 \\
1 & 0 \quad 4 \\
1 & 0
\end{pmatrix}
\text{(having minimal polynomial } (X^2 - 2)^2),
\]

and

\[
\begin{pmatrix}
0 & 2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 8 \\
1 & 0 \quad 0 \\
1 & 0 \quad -12 \\
1 & 0 \quad 0 \\
1 & 0 \quad 6 \\
1 & 0
\end{pmatrix}
\text{(having minimal polynomial } (X^2 - 2)^2),
\]

and

\[
\begin{pmatrix}
0 & 2 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 8 \\
1 & 0 \quad 0 \\
1 & 0 \quad 4 \\
1 & 0
\end{pmatrix}
\text{(having minimal polynomial } X^2 - 2).
\]

For the Jordan form of $T$, consider a base field $k$ that contains $\sqrt{2}$, and view $T$ as an endomorphism of an 8-dimensional vector space $\tilde{V}$ over $k$, so that its characteristic polynomial becomes

\[f(X) = (X - \sqrt{2})^4(X + \sqrt{2})^4.\]
Then the Jordan canonical form of the part of $T$ associated to $\sqrt{2}$ is one of

$$
\begin{pmatrix}
\sqrt{2} & \sqrt{2} \\
1 & 1 \\
\sqrt{2} & 1
\end{pmatrix}
$$
(having minimal polynomial $(X - \sqrt{2})^4$),

or

$$
\begin{pmatrix}
\sqrt{2} \\
1 \\
\sqrt{2} \\
1
\end{pmatrix}
$$
(having minimal polynomial $(X - \sqrt{2})^3$),

or

$$
\begin{pmatrix}
\sqrt{2} \\
1 \\
\sqrt{2}
\end{pmatrix}
$$
(having minimal polynomial $(X - \sqrt{2})^2$),

or

$$
\begin{pmatrix}
\sqrt{2} \\
1 \\
\sqrt{2}
\end{pmatrix}
$$
(having minimal polynomial $(X - \sqrt{2})^2$),

or

$$
\begin{pmatrix}
\sqrt{2} \\
\sqrt{2} \\
\sqrt{2} \\
\sqrt{2}
\end{pmatrix}
$$
(having minimal polynomial $X - \sqrt{2}$).

Note that the second and third cases have different minimal polynomials and yet the same geometric multiplicity of the eigenvalue $\sqrt{2}$. Note that the third and fourth cases have the same minimal polynomial and yet different geometric multiplicities of the eigenvalue $\sqrt{2}$.

Similarly for the part of $T$ associated to $-\sqrt{2}$. Thus altogether the transformation has 25 possible Jordan forms.

5. Another Example

Let $\dim_k V = 12$, and let the endomorphism $T : V \to V$ have characteristic polynomial $(X - \lambda)^{12}$ where $\lambda \in k$. Suppose we know the following nullities (kernel dimensions):

- $\mathcal{N}(T - \lambda I) = 5$,
- $\mathcal{N}((T - \lambda I)^2) = 9$,
- $\mathcal{N}((T - \lambda I)^3) = 11$,
- $\mathcal{N}((T - \lambda I)^4) = 12$.

Then the Jordan form of $T$ has five blocks, four ($9 - 5$) of which are at least 2-by-2 so that one is 1-by-1, two ($11 - 9$) of which are at least 3-by-3 so that two are
2-by-2, and one \((12 - 11)\) of which is exactly 4-by-4 so that one is 3-by-3. Thus the Jordan form of \(T\) is

\[
\begin{array}{cccc}
\lambda & & & \\
\lambda & 1 & & \\
1 & \lambda & & \\
\lambda & 1 & \lambda & \\
1 & \lambda & 1 & \\
1 & \lambda & 1 & \\
\end{array}
\]