GROUP ACTIONS

1. Review of Homomorphisms

Recall that if \((G, \circ_G)\) and \((\tilde{G}, \circ_{\tilde{G}})\) are groups then a set-map

\[ f : G \longrightarrow \tilde{G} \]

is a homomorphism if the following diagram commutes:

\[
\begin{array}{ccc}
G \times G & \xrightarrow{(f,f)} & \tilde{G} \times \tilde{G} \\
\downarrow{\circ_G} & & \downarrow{\circ_{\tilde{G}}} \\
G & \xrightarrow{f} & \tilde{G} \\
\end{array}
\]

That is, the map \(f\) must satisfy the condition

\[ f(g \circ_G g') = f(g) \circ_{\tilde{G}} f(g'), \quad g, g' \in G. \]

An injective homomorphism is a monomorphism. A surjective homomorphism is an epimorphism. A bijective homomorphism is an isomorphism. A homomorphism from a group back to itself is an endomorphism and an isomorphism from a group back to itself is an automorphism.

Immediately in consequence of the definition, any homomorphism satisfies

\[ f(e_G) = e_{\tilde{G}}, \]
\[ f(g^{-1}) = (f(g))^{-1} \quad \text{for all } g \in G, \]

and

\[ f \text{ is a monomorphism if and only if its kernel is trivial.} \]

Also, we showed in class that the inverse map of an isomorphism is again an isomorphism. That is, if a bijective set-map between groups preserves algebra then so does its inverse.

And the subgroup test quickly shows that for any homomorphism,

- \(\ker(f)\) is a subgroup of \(G\).
- \(\text{im}(f)\) is a subgroup of \(\tilde{G}\).
- \(f^{-1}(\tilde{H})\) is a subgroup of \(G\) for any subgroup \(\tilde{H}\) of \(\tilde{G}\).

If \(G\) is abelian then so is any homomorphic image \(f(G)\).

2. Group Actions

Recall also that if \(G\) is a group and \(S\) is a set then an action of \(G\) on \(S\) is a map

\[ G \times S \longrightarrow S, \quad (g, s) \longmapsto gs \]

such that

\[ es = s \quad \text{for all } s \in S, \]
\[ (gg')s = g(g's) \quad \text{for all } g, g' \in G, \ s \in S. \]
The formula \((gg')s = g(g's)\) (called the associativity rule for the action) features one group product and three group actions. The associativity rule shows immediately that
\[ g(g^{-1}s) = s, \quad g \in G, \quad s \in S. \]

3. **Isotropy**

For any group action, for any element \(s\) of the set acted on, the subset of \(G\) that fixes \(s\),
\[ G_s = \{g \in G : gs = s\}, \]
is the *isotropy subgroup* of \(s\). Verifying that \(G_s\) is indeed a subgroup is straightforward, using the last display of the previous paragraph.

4. **Application of Isotropy: Centralizing Subgroups**

Especially, any group \(G\) acts on its own power set \(\mathcal{P}(G)\) in two ways:
- By left-translation, \((g, S) \mapsto gS = \{gs : s \in S\}\).
- By left-conjugation, \((g, S) \mapsto gSg^{-1} = \{gsg^{-1} : s \in S\}\).

For any cardinal number \(k\), the two actions restrict to actions of \(G\) on the set of cardinality-\(k\) subsets of \(G\). In particular, when \(k = 1\) they restrict to actions of \(G\) on itself. The conjugation action also restricts to the set of subgroups of \(G\), and to the set of cardinality-\(k\) subgroups of \(G\) for any \(k\).

The *centralizer* of any group element \(\tilde{g}\) is defined as an isotropy subgroup,
\[ Z(\tilde{g}) = \tilde{g}\text{-isotropy under the conjugation action of } G\text{ on itself}, \]
That is, the centralizer of \(\tilde{g}\) is the subgroup of group elements that commute with \(\tilde{g}\),
\[ Z(\tilde{g}) = \{g \in G : g\tilde{g} = \tilde{g}g\}. \]
The centralizer of \(\tilde{g}\) is a supergroup of the subgroup of \(G\) generated by \(\tilde{g}\).

For any subset \(S\) of \(G\), the centralizer of \(S\) is the subgroup of group elements that commute with \(S\),
\[ Z(S) = \bigcap_{\tilde{g} \in S} Z(\tilde{g}) = \{g \in G : g\tilde{g} = \tilde{g}g \text{ for all } \tilde{g} \in S\}. \]
The centralizer of \(S\) need not contain \(S\). In particular the *center* of the group is the subgroup of elements that commute with the entire group,
\[ Z(G) = \{g \in G : g\tilde{g} = \tilde{g}g \text{ for all } \tilde{g} \in G\}. \]
A group may have abelian subgroups that are not central, since *central* connotes commuting with the entire group.

5. **Application of Isotropy: Normalizing Subgroups**

The *normalizer* of any subset \(S\) of \(G\) is its isotropy subgroup under the action of \(G\) on its power set,
\[ N(S) = \{g \in G : gSg^{-1} = S\}. \]
Elements of \(N(S)\) need not fix \(S\) pointwise under conjugation. Conjugation by elements of \(N(S)\) may permute \(S\), but it may not move elements out of \(S\).

Especially, for any homomorphism \(f : G \rightarrow \tilde{G}\),
\[ N(\ker(f)) = G. \]
Indeed, for any $k \in \ker(f)$ and any $g \in G$,
\[
f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)e_G(f(g))^{-1} = e_G.
\]

For another example in the same spirit, consider some subgroups of $\text{GL}_2(F)$ where $F$ is any field,

\[
P = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\} \quad \text{(the parabolic subgroup)},
\]
\[
M = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\} \quad \text{(the maximal Levi component)},
\]
\[
N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right\} \quad \text{(the unipotent radical)}.
\]

The calculation
\[
\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}
\]
shows that
\[
P = MN = NM.
\]

Also, the intersection $M \cap N$ is trivial. And, although $M$ and $N$ do not commute, $M$ normalizes $N$,

\[
mnm^{-1} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix} = \begin{bmatrix} 1 & abd^{-1} \\ 0 & 1 \end{bmatrix} \in N.
\]

That is, the normalizer of $N$ in $P$ is all of $P$. On the other hand, one can check that $N$ does not normalize $M$. 