

STEREOGRAPHIC PROJECTION IS CONFORMAL

Let

$$S^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$$

be the unit sphere, and let \mathbf{n} denote the north pole $(0, 0, 1)$. Identify the complex plane \mathbf{C} with the (x, y) -plane in \mathbf{R}^3 . The *stereographic projection* map,

$$\pi : S^2 - \mathbf{n} \rightarrow \mathbf{C},$$

is described as follows: place a light source at the north pole \mathbf{n} . For any point $p \in S^2 - \mathbf{n}$, consider a light ray emanating downward from \mathbf{n} to pass through the sphere at p . The ray also meets the plane, and the point where it hits is $\pi(p)$. That is,

$$\pi(p) = \ell(\mathbf{n}, p) \cap \mathbf{R}^2,$$

where $\ell(\mathbf{n}, p) = \{(1-t)\mathbf{n} + tp : t \in \mathbf{R}\}$ is the line through \mathbf{n} and p . (See figure 1.)

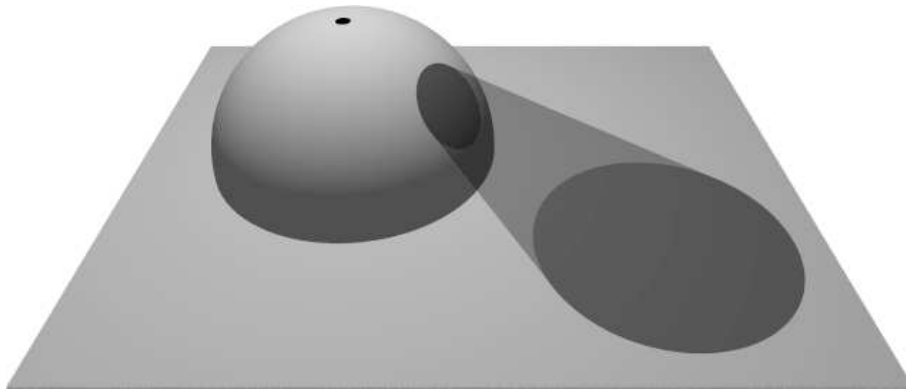


FIGURE 1. Stereographic projection

The formula for stereographic projection is

$$\pi(x, y, z) = \frac{x + iy}{1 - z}.$$

Indeed, the point $(1-t)\mathbf{n} + t(x, y, z)$ has last coordinate $1-t+tz$. This equals 0 for $t = 1/(1-z)$, making the other coordinates $x/(1-z)$ and $y/(1-z)$, and the formula follows.

For the inverse map, take a point $q = (x, y, 0)$ in the plane. Since \mathbf{n} and q are orthogonal, any point $p = (1-t)\mathbf{n} + tq$ on the line $\ell(\mathbf{n}, q)$ satisfies $|p|^2 = (1-t)^2 + t^2|q|^2$. This equals 1 for $t = 2/(|q|^2 + 1)$ (ignoring $t = 0$, which gives the north pole), showing that

$$\pi^{-1}(x, y) = \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right).$$

Stereographic projection is conformal, meaning that it preserves angles between curves. To see this, take a point $p \in S^2 \setminus \{\mathbf{n}\}$, let T_p denote the tangent plane to S^2 at p , and let $T_{\mathbf{n}}$ denote the tangent plane to S^2 at \mathbf{n} . Working first in the \mathbf{Onp} -plane (see figure 2), we have equal angles α and right angles between the radii and the tangent planes, hence equal angles β , hence equal angles β' , and hence equal lengths b .

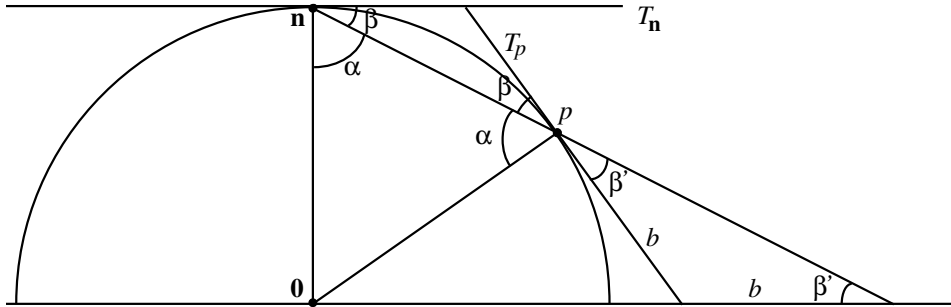


FIGURE 2. Side view of stereographic projection

Now let γ be a smooth curve on S^2 through p , let t be its tangent at p , and let \hat{t} be the intersection of the plane containing \mathbf{n} and t with \mathbf{R}^2 . (See figure 3.) In fact \hat{t} is the tangent to $\pi \circ \gamma$ at $\pi(p)$. To see this, note that π is the restriction of a rational, hence differentiable, map (also called π) from an \mathbf{R}^3 -neighborhood of p to \mathbf{R}^2 that takes t to \hat{t} near p . (A *neighborhood* of a point is an open set containing the point.) Since γ and t are curves in \mathbf{R}^3 with the same tangent t at p , it follows that $\pi \circ \gamma$ and $\pi \circ t = \hat{t}$ are curves in \mathbf{R}^2 with the same tangent at $\pi(p)$. Since \hat{t} is its own tangent at $\pi(p)$, it is also the tangent to $\pi \circ \gamma$ there. The lengths b are equal, hence so are the angles θ , by right triangles. Repeating this analysis for a second curve $\tilde{\gamma}$ through p completes the proof.

For a continuation of this argument, showing that stereographic projection takes circles to circles, see *Geometry and the Imagination* by Hilbert and Cohn-Vossen.

FOLLOWUP EXERCISES

- Illustrate the proof that stereographic projection is conformal when p lies in the lower hemisphere.
- The proof that stereographic projection is conformal tacitly assumed that t and \hat{t} meet. Must they? What happens to the proof if they don't?
- Show that stereographic projection takes circles to circles.

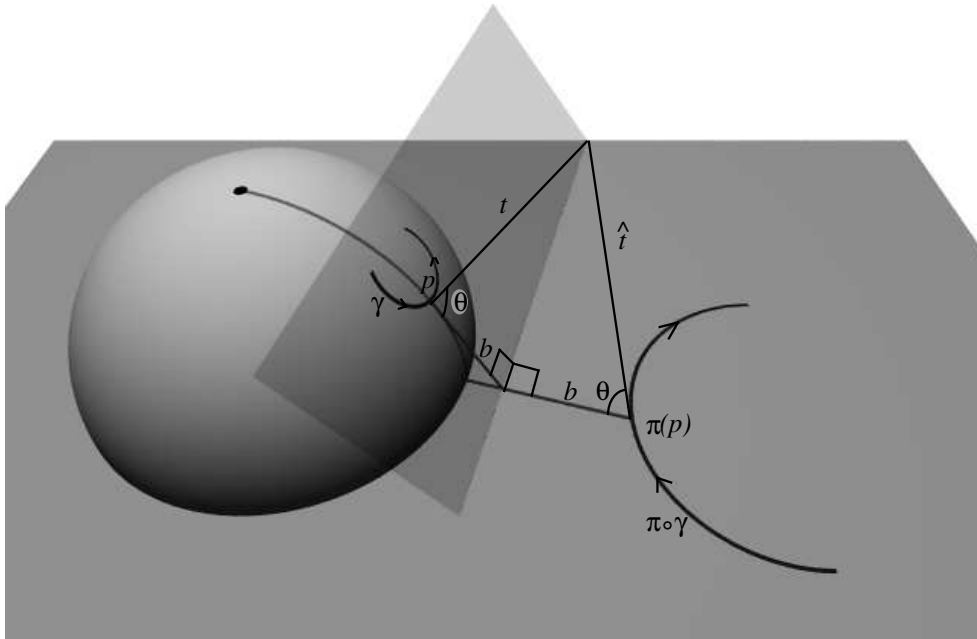


FIGURE 3. Stereographic projection is conformal