BASIC SYMBOL-PATTERNS FOR SETS AND MAPS

Let $X$ and $Y$ be sets, assuming no further structure whatsoever. Let
\[ f : X \rightarrow Y \]
be a function, with $Y = f(X)$ but subject to no other assumptions. Let $I$ be any
index set. Let
\[ S, \{S_i : i \in I\} \]
be arbitrary subsets of $X$, and let
\[ T, \{T_i : i \in I\} \]
be arbitrary subsets of $Y$. As usual, let $f^{-1}$ denote inverse image, and let $^c$ denote complementation.

Each of the following eight pairs is plausibly related:
\[
\begin{align*}
&f\left(\bigcup S_i\right) \text{ and } \bigcup f(S_i), & f^{-1}\left(\bigcup T_i\right) \text{ and } \bigcup f^{-1}(T_i), \\
&f\left(\bigcap S_i\right) \text{ and } \bigcap f(S_i), & f^{-1}\left(\bigcap T_i\right) \text{ and } \bigcap f^{-1}(T_i), \\
&f(S^c) \text{ and } (f(S))^c, & f^{-1}(T^c) \text{ and } (f^{-1}(T))^c, \\
&f^{-1}(f(S)) \text{ and } S, & f(f^{-1}(T)) \text{ and } T.
\end{align*}
\]
For each pair, determine whether equality must hold, and if it doesn’t, whether a
containment must hold. When a containment holds, what conditions force it to be
equality?