ROTATIONS OF THE RIEMANN SPHERE

A rotation of the sphere $S^2$ is a map $r = r_{p,\alpha}$ described by spinning the sphere (actually, spinning the ambient space $\mathbb{R}^3$) about the line through the origin and the point $p \in S^2$, counterclockwise through angle $\alpha$ looking at $p$ from outside the sphere. (See figure 1.)

![Figure 1](image.png)

**Figure 1.** The rotation $r_{p,\alpha}$

Thus $r$ is the linear map that fixes $p$ and rotates planes orthogonal to $p$ through angle $\alpha$. Let $q$ be a unit vector orthogonal to $p$. Then the matrix of $r$ is (viewing $p$ and $q$ as column vectors)

$$
m_r = [p \ q \ p \times q] \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix} [p \ q \ p \times q]^{-1}.
$$

The set $\text{Rot}(S^2)$ of such rotations forms a group, most naturally viewed as a subgroup of $\text{GL}_3(\mathbb{R})$. Showing this requires some linear algebra.

Recall that if $m \in \text{M}_3(\mathbb{R})$, meaning that $m$ is a 3-by-3 real matrix, then its transpose $m^t$ is obtained by flipping about the diagonal. That is,

$$m^t_{ij} = m_{ji} \quad \text{for } i, j = 1, 2, 3.$$

The transpose is characterized by the more convenient condition

$$\langle mx, y \rangle = \langle x, m^t y \rangle \quad \text{for all } x, y \in \mathbb{R}^3,$$

where $\langle \cdot , \cdot \rangle$ is the usual inner product,

$$\langle x, y \rangle = \sum x_i y_i.$$

The matrix $m$ is orthogonal if

$$m^t m = I,$$

or, equivalently, if $m$ preserves inner products,

$$\langle mx, my \rangle = \langle x, y \rangle \quad \text{for all } x, y \in \mathbb{R}^3.$$
The orthogonal matrices form a group $O_3(\mathbb{R}) \subset \text{GL}_3(\mathbb{R})$, and the special orthogonal matrices,

$$SO_3(\mathbb{R}) = \{ m \in O_3(\mathbb{R}) : \det m = 1 \},$$

form a subgroup of index 2. With these facts in place it is not hard to prove that $\text{Rot}(S^2)$ forms a group, and that

**Theorem 0.1.** As a subgroup of $\text{GL}_3(\mathbb{R})$, $\text{Rot}(S^2) = SO_3(\mathbb{R})$.

Here is a sketch of the proof. Given a rotation $r = r_{p,\alpha}$, its matrix,

$$m_r = \begin{bmatrix} p & q & p \times q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} p & q & p \times q \end{bmatrix}^{-1},$$

is readily verified to be special orthogonal. On the other hand, take any special orthogonal matrix $m$. Since 3 is odd, $m$ has a real eigenvalue $\lambda$. Any real eigenvalue $\lambda$ with eigenvector $p$ satisfies

$$\lambda^2 \langle p, p \rangle = \langle \lambda p, \lambda p \rangle = \langle mp, mp \rangle = \langle p, p \rangle,$$

i.e., $\lambda = \pm 1$. Since $\det m = 1$, and the determinant is the product of the eigenvalues, and any imaginary eigenvalues occur in conjugate pairs, $m$ in fact has 1 for an eigenvalue with unit eigenvector $p$. Take any nonzero vector $q$ perpendicular to $p$. Some rotation $r = r_{p,\alpha}$ takes $q$ to $mq$ and has matrix $m_r \in SO_3(\mathbb{R})$. Thus the matrix $m_r^{-1}m$ lies in $SO_3(\mathbb{R})$ and fixes both $p$ and $q$. It is therefore the identity, showing that $m = m_r$ is a rotation matrix.

A rotation of the Riemann sphere $\hat{\mathbb{C}}$ is a map $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ corresponding under stereographic projection to a true rotation $r$ of the round sphere $S^2$. In other words, the following diagram commutes:

$$
\begin{array}{ccc}
S^2 & \xrightarrow{r} & S^2 \\
\pi \downarrow & & \downarrow \pi \\
\hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}}.
\end{array}
$$

Let $\text{Rot}(\hat{\mathbb{C}})$ denote the set of such rotations. Since $\text{Rot}(S^2)$ forms a group, $\text{Rot}(\hat{\mathbb{C}})$ forms an isomorphic group under $r \mapsto \pi \circ r \circ \pi^{-1}$. Since any rotation $r$ is conformal on $S^2$, the corresponding bijection $f$ is conformal on $\hat{\mathbb{C}}$ and is therefore an automorphism, and so $\text{Rot}(\hat{\mathbb{C}})$ is a subgroup of $\text{Aut}(\hat{\mathbb{C}})$. With some more linear algebra we can describe $\text{Rot}(\hat{\mathbb{C}})$ explicitly as a subgroup of $\text{PSL}_2(\mathbb{C})$.

If $m \in M_2(\mathbb{C})$ is a 2-by-2 complex matrix then its adjoint is

$$m^* = \overline{m'},$$

where the overbar denotes complex conjugation, i.e.,

$$m^*_{ij} = \overline{m'_{ji}} \quad \text{for } i, j = 1, 2.$$

The adjoint is characterized by the condition

$$\langle mx, y \rangle = \langle x, m^*y \rangle \quad \text{for all } x, y \in \mathbb{C}^2,$$
where now $\langle \ , \ \rangle$ is the complex inner product
$$\langle x, y \rangle = \sum x_i y_i.$$ The role of the adjoint in the algebra of complex matrices is analogous to the role of the conjugate in the algebra of complex numbers. The matrix $u$ is unitary if
$$u^* u = I.$$ (This condition generalizes the unit complex numbers.) Equivalently,
$$\langle u x, u y \rangle = \langle x, y \rangle$$ for all $x, y \in \mathbb{C}^2$.

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The unitary matrices form a group $U_2(\mathbb{C})$. The special unitary matrices $SU_2(\mathbb{C}) = \{ u \in U_2(\mathbb{C}) : \det u = 1 \}$ form a subgroup. A matrix is special unitary if and only if it takes the form
$$u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad |a|^2 + |b|^2 = 1.$$ The projective unitary group is
$$PU_2(\mathbb{C}) = U_2(\mathbb{C})/(U_2(\mathbb{C}) \cap \mathbb{C}^* I),$$ and the projective special unitary group is
$$PSU_2(\mathbb{C}) = SU_2(\mathbb{C})/(SU_2(\mathbb{C}) \cap \mathbb{C}^* I) = SU_2(\mathbb{C})/\{\pm I\}.$$ There is an isomorphism $PU_2(\mathbb{C}) \cong PSU_2(\mathbb{C})$, and the group $PSU_2(\mathbb{C})$ can be more convenient to work with since its elements are two-element cosets $\{\pm u\}$.

**Theorem 0.2.** As a subgroup of $PSL_2(\mathbb{C})$, $Rot(\hat{\mathbb{C}}) = PSU_2(\mathbb{C})$.

Here is an elegant proof, which incidentally shows that $Rot(\hat{\mathbb{C}})$ is a group without reference to $SO_3(\mathbb{R})$. We show first that any rotation lies in $PSU_2(\mathbb{C})$, second that any element of $PSU_2(\mathbb{C})$ is a rotation.

A short calculation shows that if the antipodal pair $p, -p \in S^2 \setminus \{n, s\}$ have stereographic images $z, z^* \in \mathbb{C}$, then $z^* = -1/\bar{z}$, where the overbar is complex conjugation. Now let $f$ be a rotation of $\hat{\mathbb{C}}$ induced by a rotation $r$ of $S^2$. Let a matrix describing $f$ be
$$m_f = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad \det(m_f) = 1.$$ Since $r$ takes antipodal pairs to antipodal pairs, $f$ must satisfy the corresponding relation
$$f(z^*) = f(z)^*$$ for all $z \in \mathbb{C} \setminus \{0\}$.

This condition is that for some $\lambda \in \mathbb{C}^*$,
$$d = \lambda \bar{a}, \quad a = \lambda d, \quad c = -\lambda \bar{b}, \quad b = -\lambda \bar{c}.$$ These relations and the relation $ad - bc = 1$ combine to show that $\lambda = 1$ and therefore $m_f \in PSU_2(\mathbb{C})$.

For the converse, let $f$ have matrix
$$m_f = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in PSU_2(\mathbb{C}).$$ If $f(0) = 0$ then $f(z) = e^{i\alpha}z$ for some $\alpha$, so $f$ is a rotation. If $f(0) = z \neq 0$ then some rotation $f_z \in Rot(\hat{\mathbb{C}}) \subset PSU_2(\mathbb{C})$ also takes $0$ to $z$, and so the composition
Theorem 0.3. Let \( f \in \text{PSU}_2(\mathbb{C}) \) fixes 0 and is thus a rotation. Therefore \( f = f_z \circ g \) is also a rotation, and the proof is complete.

The two theorems combine to show that
\[
\text{PSU}_2(\mathbb{C}) \cong \text{SO}_3(\mathbb{R}).
\]

The next result says how to compute in \( \text{PSU}_2(\mathbb{C}) \) while thinking of Rot(\( S^2 \)). For any rotation \( r_{p,\alpha} \) of \( S^2 \), let \( f_{\pi(p),\alpha} \) denote the corresponding rotation of \( \hat{\mathbb{C}} \).

**Theorem 0.3.** Let \( p = (p_1, p_2, p_3) \in S^2 \) and let \( \alpha \in \mathbb{R} \). Then
\[
f_{\pi(p),\alpha} = \begin{bmatrix}
\cos \frac{\alpha}{2} + ip_3 \sin \frac{\alpha}{2} & -p_2 \sin \frac{\alpha}{2} + ip_1 \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} - ip_3 \sin \frac{\alpha}{2}
\end{bmatrix}.
\]

Here is the proof. Either by geometry or by a calculation using the commutative diagram from earlier, the rotation \( r_{n,\alpha} \) of \( S^2 \) induces the automorphism \( f_{\infty,\alpha}(z) = e^{i\alpha}z \) of \( \hat{\mathbb{C}} \), i.e., under a slight abuse of notation,
\[
f_{\infty,\alpha} = \begin{bmatrix}
e^{i\alpha/2} & 0 \\
0 & e^{-i\alpha/2}
\end{bmatrix}.
\]

Next consider the rotation \( r_{(0,1,0),\phi} \) of \( S^2 \) counterclockwise about the positive \( x_2 \)-axis through angle \( \phi \). We will find the corresponding rotation \( f_{i,\phi} \) of \( \hat{\mathbb{C}} \). A rotation \( r \) of \( S^2 \) takes \((0,1,0)\) to \( n \) and \((0,-1,0)\) to \( s \); the corresponding rotation \( f \) of \( \hat{\mathbb{C}} \) takes \( i \) to \( \infty \) and \(-i\) to 0, so it takes the form
\[
f(z) = k \frac{z + i}{z - i}
\]
for some nonzero constant \( k \). Since \( r_{(0,1,0),\phi} = r^{-1} \circ r_{n,\phi} \circ r \), the corresponding result in Rot(\( \hat{\mathbb{C}} \)) is
\[
f_{i,\phi} = f^{-1} \circ f_{\infty,\phi} \circ f,
\]
or
\[
f \circ f_{i,\phi} = f_{\infty,\phi} \circ f.
\]

Thus for all \( z \in \hat{\mathbb{C}} \),
\[
k \cdot \frac{f_{i,\phi}(z) + i}{f_{i,\phi}(z) - i} = e^{i\phi/2} \frac{z + i}{z - i}.
\]
The \( k \) cancels, leaving
\[
e^{-i\phi/2}(f_{i,\phi}(z) + i)(z - i) = e^{i\phi/2}(f_{i,\phi}(z) - i)(z + i),
\]
and some algebra gives
\[
f_{i,\phi} = \begin{bmatrix}
\cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\
\sin \frac{\phi}{2} & \cos \frac{\phi}{2}
\end{bmatrix}.
\]

Now let the point \( p \in S^2 \) have spherical coordinates \( (1, \theta, \phi) \), meaning that
\[
\cos \theta = p_1/\sqrt{p_1^2 + p_2^2}, \quad \sin \theta = p_2/\sqrt{p_1^2 + p_2^2}, \quad \cos \phi = p_3, \quad \sin \phi = \sqrt{p_1^2 + p_2^2}.
\]
(See figure 2.) To carry out \( r_{p,\alpha} \), move \( p \) to the north pole via rotations about the north pole and \((0,1,0)\), rotate about the north pole by \( \alpha \), and restore \( p \); to wit,
\[
r_{p,\alpha} = r_{n,\theta} \circ r_{(0,1,0),\phi} \circ r_{n,\alpha} \circ r_{(0,1,0),-\phi} \circ r_{n,-\theta}.
\]
The corresponding rotation of $\hat{C}$ is

$$f_{\pi(p),\alpha} = \left[ e^{i\theta/2} 0 \right] \left[ \begin{array}{cc} \cos \phi & -e^{-i\theta/2} \\ \sin \phi & e^{-i\theta/2} \end{array} \right] \left[ \begin{array}{cc} e^{i\alpha/2} 0 \\ 0 e^{-i\alpha/2} \end{array} \right]$$

$$\cdot \left[ \begin{array}{cc} \cos \frac{\phi}{2} & \sin \frac{\phi}{2} \\ -\sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{array} \right] \left[ \begin{array}{cc} e^{-i\theta/2} 0 \\ 0 e^{i\theta/2} \end{array} \right].$$

Multiplying this out and using a little trigonometry gives the result.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{spherical_coordinates}
\caption{Spherical coordinates}
\end{figure}