ROTATIONS OF THE RIEMANN SPHERE

A *rotation* of the sphere S^2 is a map $r = r_{p,\alpha}$ described by spinning the sphere (actually, spinning the ambient space $\mathbb{R}^3$) about the line through the origin and the point $p \in S^2$, counterclockwise through angle α looking at p from outside the sphere. (See figure 1.)

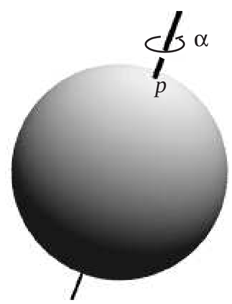

FIGURE 1. The rotation $r_{p,\alpha}$

Thus r is the linear map that fixes p and rotates planes orthogonal to p through angle α. Let q be a unit vector orthogonal to p. Then the matrix of r is (viewing p and q as column vectors)

$$m_r = \begin{bmatrix} p & q & p \times q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} p & q & p \times q \end{bmatrix}^{-1}.$$

The set

$$\mathrm{Rot}(S^2)$$

of such rotations forms a group, most naturally viewed as a subgroup of $\mathrm{GL}_3(\mathbb{R})$. Showing this requires some linear algebra.

Recall that if $m \in \mathrm{M}_3(\mathbb{R})$, meaning that m is a 3-by-3 real matrix, then its *transpose* m^t is obtained by flipping about the diagonal. That is,

$$m^t_{ij} = m_{ji} \quad \text{for } i, j = 1, 2, 3.$$

The transpose is characterized by the more convenient condition

$$\langle mx, y \rangle = \langle x, m^t y \rangle \quad \text{for all } x, y \in \mathbb{R}^3,$$

where $\langle\ ,\ \rangle$ is the usual inner product,

$$\langle x, y \rangle = \sum x_i y_i.$$

The matrix m is *orthogonal* if

$$m^t m = I,$$

or, equivalently, if m preserves inner products,

$$\langle mx, my \rangle = \langle x, y \rangle \quad \text{for all } x, y \in \mathbb{R}^3.$$

The orthogonal matrices form a group $\mathrm{O}_3(\mathbb{R}) \subset \mathrm{GL}_3(\mathbb{R})$, and the special orthogonal matrices,

$$\mathrm{SO}_3(\mathbb{R}) = \{m \in \mathrm{O}_3(\mathbb{R}) : \det m = 1\},$$

form a subgroup of index 2. With these facts in place it is not hard to prove that $\mathrm{Rot}(S^2)$ forms a group, and that

Theorem 0.1. *As a subgroup of* $\mathrm{GL}_3(\mathbb{R})$, $\mathrm{Rot}(S^2) = \mathrm{SO}_3(\mathbb{R})$.

Here is a sketch of the proof. Given a rotation $r = r_{p,\alpha}$, its matrix,

$$m_r = \begin{bmatrix} p & q & p \times q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} p & q & p \times q \end{bmatrix}^{-1},$$

is readily verified to be special orthogonal. On the other hand, take any special orthogonal matrix m. Since 3 is odd, m has a real eigenvalue λ. Any real eigenvalue λ with eigenvector p satisfies

$$\lambda^2 \langle p, p \rangle = \langle \lambda p, \lambda p \rangle = \langle mp, mp \rangle = \langle p, p \rangle,$$

i.e., $\lambda = \pm 1$. Since $\det m = 1$, and the determinant is the product of the eigenvalues, and any imaginary eigenvalues occur in conjugate pairs, m in fact has 1 for an eigenvalue with unit eigenvector p. Take any nonzero vector q perpendicular to p. Some rotation $r = r_{p,\alpha}$ takes q to mq and has matrix $m_r \in \mathrm{SO}_3(\mathbb{R})$. Thus the matrix $m_r^{-1} m$ lies in $\mathrm{SO}_3(\mathbb{R})$ and fixes both p and q. It is therefore the identity, showing that $m = m_r$ is a rotation matrix.

A *rotation* of the Riemann sphere $\widehat{\mathbb{C}}$ is a map $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ corresponding under stereographic projection to a true rotation r of the round sphere S^2. In other words, the following diagram commutes:

$$\begin{tikzcd} S^2 \arrow[r, "r"] \arrow[d, "\pi"'] & S^2 \arrow[d, "\pi"] \\ \widehat{\mathbb{C}} \arrow[r, "f"] & \widehat{\mathbb{C}}. \end{tikzcd}$$

Let

$$\mathrm{Rot}(\widehat{\mathbb{C}})$$

denote the set of such rotations. Since $\mathrm{Rot}(S^2)$ forms a group, $\mathrm{Rot}(\widehat{\mathbb{C}})$ forms an isomorphic group under $r \mapsto \pi \circ r \circ \pi^{-1}$. Since any rotation r is conformal on S^2, the corresponding bijection f is conformal on $\widehat{\mathbb{C}}$ and is therefore an automorphism, and so $\mathrm{Rot}(\widehat{\mathbb{C}})$ is a subgroup of $\mathrm{Aut}(\widehat{\mathbb{C}})$. With some more linear algebra we can describe $\mathrm{Rot}(\widehat{\mathbb{C}})$ explicitly as a subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

If $m \in \mathrm{M}_2(\mathbb{C})$ is a 2-by-2 complex matrix then its *adjoint* is

$$m^* = \overline{m}^t,$$

where the overbar denotes complex conjugation, i.e.,

$$m^*_{ij} = \overline{m_{ji}} \quad \text{for } i, j = 1, 2.$$

The adjoint is characterized by the condition

$$\langle mx, y \rangle = \langle x, m^* y \rangle \quad \text{for all } x, y \in \mathbb{C}^2,$$

where now $\langle\ ,\ \rangle$ is the complex inner product

$$\langle x, y\rangle = \sum \overline{x_i} y_i.$$

The role of the adjoint in the algebra of complex matrices is analogous to the role of the conjugate in the algebra of complex numbers. The matrix u is *unitary* if

$$u^* u = I.$$

(This condition generalizes the unit complex numbers.) Equivalently,

$$\langle ux, uy\rangle = \langle x, y\rangle \quad \text{for all } x, y \in \mathbb{C}^2.$$

The unitary matrices form a group $\mathrm{U}_2(\mathbb{C})$. The special unitary matrices

$$\mathrm{SU}_2(\mathbb{C}) = \{u \in \mathrm{U}_2(\mathbb{C}) : \det u = 1\}$$

form a subgroup. A matrix is special unitary if and only if it takes the form

$$u = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}, \quad |a|^2 + |b|^2 = 1.$$

The projective unitary group is

$$\mathrm{PU}_2(\mathbb{C}) = \mathrm{U}_2(\mathbb{C})/(\mathrm{U}_2(\mathbb{C}) \cap \mathbb{C}^* I),$$

and the projective special unitary group is

$$\mathrm{PSU}_2(\mathbb{C}) = \mathrm{SU}_2(\mathbb{C})/(\mathrm{SU}_2(\mathbb{C}) \cap \mathbb{C}^* I) = \mathrm{SU}_2(\mathbb{C})/\{\pm I\}.$$

There is an isomorphism $\mathrm{PU}_2(\mathbb{C}) \cong \mathrm{PSU}_2(\mathbb{C})$, and the group $\mathrm{PSU}_2(\mathbb{C})$ can be more convenient to work with since its elements are two-element cosets $\{\pm u\}$.

Theorem 0.2. *As a subgroup of* $\mathrm{PSL}_2(\mathbb{C})$*,* $\mathrm{Rot}(\widehat{\mathbb{C}}) = \mathrm{PSU}_2(\mathbb{C})$*.*

Here is an elegant proof, which incidentally shows that $\mathrm{Rot}(\widehat{\mathbb{C}})$ is a group without reference to $\mathrm{SO}_3(\mathbb{R})$. We show first that any rotation lies in $\mathrm{PSU}_2(\mathbb{C})$, second that any element of $\mathrm{PSU}_2(\mathbb{C})$ is a rotation.

A short calculation shows that if the antipodal pair $p, -p \in S^2 \setminus \{\mathbf{n}, \mathbf{s}\}$ have stereographic images $z, z^* \in \mathbb{C}$, then $z^* = -1/\overline{z}$, where the overbar is complex conjugation. Now let f be a rotation of $\widehat{\mathbb{C}}$ induced by a rotation r of S^2. Let a matrix describing f be

$$m_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(m_f) = 1.$$

Since r takes antipodal pairs to antipodal pairs, f must satisfy the corresponding relation

$$f(z^*) = f(z)^* \quad \text{for all } z \in \mathbb{C} \setminus \{0\}.$$

This condition is that for some $\lambda \in \mathbb{C}^*$,

$$d = \lambda\overline{a}, \quad a = \lambda\overline{d}, \quad c = -\lambda\overline{b}, \quad b = -\lambda\overline{c}.$$

These relations and the relation $ad - bc = 1$ combine to show that $\lambda = 1$ and therefore $m_f \in \mathrm{PSU}_2(\mathbb{C})$.

For the converse, let f have matrix

$$m_f = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \in \mathrm{PSU}_2(\mathbb{C}).$$

If $f(0) = 0$ then $f(z) = e^{i\alpha} z$ for some α, so f is a rotation. If $f(0) = z \neq 0$ then some rotation $f_z \in \mathrm{Rot}(\widehat{\mathbb{C}}) \subset \mathrm{PSU}_2(\mathbb{C})$ also takes 0 to z, and so the composition

$g = f_z^{-1} \circ f \in \mathrm{PSU}_2(\mathbb{C})$ fixes 0 and is thus a rotation. Therefore $f = f_z \circ g$ is also a rotation, and the proof is complete.

The two theorems combine to show that

$$\mathrm{PSU}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{R}).$$

The next result says how to compute in $\mathrm{PSU}_2(\mathbb{C})$ while thinking of $\mathrm{Rot}(S^2)$. For any rotation $r_{p,\alpha}$ of S^2, let $f_{\pi(p),\alpha}$ denote the corresponding rotation of $\widehat{\mathbb{C}}$.

Theorem 0.3. *Let $p = (p_1, p_2, p_3) \in S^2$ and let $\alpha \in \mathbb{R}$. Then*

$$f_{\pi(p),\alpha} = \begin{bmatrix} \cos\frac{\alpha}{2} + ip_3 \sin\frac{\alpha}{2} & -p_2 \sin\frac{\alpha}{2} + ip_1 \sin\frac{\alpha}{2} \\ p_2 \sin\frac{\alpha}{2} + ip_1 \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} - ip_3 \sin\frac{\alpha}{2} \end{bmatrix}.$$

Here is the proof. Either by geometry or by a calculation using the commutative diagram from earlier, the rotation $r_{\mathbf{n},\alpha}$ of S^2 induces the automorphism $f_{\infty,\alpha}(z) = e^{i\alpha}z$ of $\widehat{\mathbb{C}}$, i.e., under a slight abuse of notation,

$$f_{\infty,\alpha} = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}.$$

Next consider the rotation $r_{(0,1,0),\phi}$ of S^2 counterclockwise about the positive x_2-axis through angle ϕ. We will find the corresponding rotation $f_{i,\phi}$ of $\widehat{\mathbb{C}}$. A rotation r of S^2 takes $(0,1,0)$ to $\mathbf{n}$ and $(0,-1,0)$ to $\mathbf{s}$; the corresponding rotation f of $\widehat{\mathbb{C}}$ takes i to ∞ and $-i$ to 0, so it takes the form

$$f(z) = k\frac{z+i}{z-i}$$

for some nonzero constant k. Since $r_{(0,1,0),\phi} = r^{-1} \circ r_{\mathbf{n},\phi} \circ r$, the corresponding result in $\mathrm{Rot}(\widehat{\mathbb{C}})$ is

$$f_{i,\phi} = f^{-1} \circ f_{\infty,\phi} \circ f,$$

or

$$f \circ f_{i,\phi} = f_{\infty,\phi} \circ f.$$

Thus for all $z \in \widehat{\mathbb{C}}$,

$$k \cdot \frac{f_{i,\phi}(z) + i}{f_{i,\phi}(z) - i} = e^{i\phi}k \cdot \frac{z+i}{z-i}.$$

The k cancels, leaving

$$e^{-i\phi/2}(f_{i,\phi}(z) + i)(z - i) = e^{i\phi/2}(f_{i,\phi}(z) - i)(z + i),$$

and some algebra gives

$$f_{i,\phi} = \begin{bmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}.$$

Now let the point $p \in S^2$ have spherical coordinates $(1, \theta, \phi)$, meaning that

$$\cos\theta = p_1/\sqrt{p_1^2 + p_2^2}, \quad \sin\theta = p_2/\sqrt{p_1^2 + p_2^2}, \quad \cos\phi = p_3, \quad \sin\phi = \sqrt{p_1^2 + p_2^2}.$$

(See figure 2.) To carry out $r_{p,\alpha}$, move p to the north pole via rotations about the north pole and $(0,1,0)$, rotate about the north pole by α, and restore p; to wit,

$$r_{p,\alpha} = r_{\mathbf{n},\theta} \circ r_{(0,1,0),\phi} \circ r_{\mathbf{n},\alpha} \circ r_{(0,1,0),-\phi} \circ r_{\mathbf{n},-\theta}.$$

The corresponding rotation of $\widehat{\mathbb{C}}$ is

$$f_{\pi(p),\alpha} = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix} \cdot \begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

Multiplying this out and using a little trigonometry gives the result.

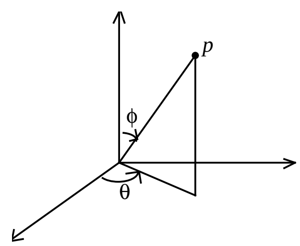

FIGURE 2. Spherical coordinates