ROTATIONS OF THE RIEMANN SPHERE

A rotation of the sphere $S^2$ is a map $r = r_{p,\alpha}$ described by spinning the sphere (actually, spinning the ambient space $\mathbb{R}^3$) about the line through the origin and the point $p \in S^2$, counterclockwise through angle $\alpha$ looking at $p$ from outside the sphere. (See figure 1.)

![Figure 1. The rotation $r_{p,\alpha}$]

Thus $r$ is the linear map that fixes $p$ and rotates planes orthogonal to $p$ through angle $\alpha$. Let $q$ be a unit vector orthogonal to $p$. Then the matrix of $r$ is (viewing $p$ and $q$ as column vectors)

$$m_r = \begin{bmatrix} p & q & p \times q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} p & q & p \times q \end{bmatrix}^{-1}.$$

The set

$$\text{Rot}(S^2)$$

of such rotations forms a group, most naturally viewed as a subgroup of $\text{GL}_3(\mathbb{R})$. Showing this requires some linear algebra.

Recall that if $m \in M_3(\mathbb{R})$, meaning that $m$ is a 3-by-3 real matrix, then its transpose $m^t$ is obtained by flipping about the diagonal. That is,

$$m^t_{ij} = m_{ji} \text{ for } i, j = 1, 2, 3.$$

The transpose is characterized by the more convenient condition

$$\langle mx, y \rangle = \langle x, m^t y \rangle \text{ for all } x, y \in \mathbb{R}^3,$$

where $\langle , \rangle$ is the usual inner product,

$$\langle x, y \rangle = \sum x_i y_i.$$

The matrix $m$ is orthogonal if

$$m^t m = I,$$

or, equivalently, if $m$ preserves inner products,

$$\langle mx, my \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{R}^3.$$
The orthogonal matrices form a group $O_3(\mathbb{R}) \subset GL_3(\mathbb{R})$, and the special orthogonal matrices,

$$SO_3(\mathbb{R}) = \{ m \in O_3(\mathbb{R}) : \det m = 1 \},$$

form a subgroup of index 2. With these facts in place it is not hard to prove that $\text{Rot}(S^2)$ forms a group, and that

**Theorem 0.1.** As a subgroup of $GL_3(\mathbb{R})$, $\text{Rot}(S^2) = SO_3(\mathbb{R})$.

Here is a sketch of the proof. Given a rotation $r = r_{p, \alpha}$, its matrix,

$$m_r = \begin{bmatrix} p & q & p \times q \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} p & q & p \times q \end{bmatrix}^{-1},$$

is readily verified to be special orthogonal. On the other hand, take any special orthogonal matrix $m$. Since 3 is odd, $m$ has a real eigenvalue $\lambda$. Any real eigenvalue $\lambda$ with eigenvector $p$ satisfies

$$\lambda^2(p, p) = \langle \lambda p, \lambda p \rangle = \langle mp, mp \rangle = \langle p, p \rangle,$$

i.e., $\lambda = \pm 1$. Since $\det m = 1$, and the determinant is the product of the eigenvalues, and any imaginary eigenvalues occur in conjugate pairs, $m$ in fact has 1 for an eigenvalue with unit eigenvector $p$. Take any nonzero vector $q$ perpendicular to $p$. Some rotation $r = r_{p,\alpha}$ takes $q$ to $mq$ and has matrix $m_r \in SO_3(\mathbb{R})$. Thus the matrix $m_r^{-1}m$ lies in $SO_3(\mathbb{R})$ and fixes both $p$ and $q$. It is therefore the identity, showing that $m = m_r$ is a rotation matrix.

A rotation of the Riemann sphere $\hat{\mathbb{C}}$ is a map $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ corresponding under stereographic projection to a true rotation $r$ of the round sphere $S^2$. In other words, the following diagram commutes:

$$\begin{array}{ccc}
S^2 & \xrightarrow{r} & S^2 \\
\pi \downarrow & & \downarrow \pi \\
\hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}}.
\end{array}$$

Let

$$\text{Rot}(\hat{\mathbb{C}})$$

denote the set of such rotations. Since $\text{Rot}(S^2)$ forms a group, $\text{Rot}(\hat{\mathbb{C}})$ forms an isomorphic group under $r \mapsto \pi \circ r \circ \pi^{-1}$. Since any rotation $r$ is conformal on $S^2$, the corresponding bijection $f$ is conformal on $\hat{\mathbb{C}}$ and is therefore an automorphism, and so $\text{Rot}(\hat{\mathbb{C}})$ is a subgroup of $\text{Aut}(\hat{\mathbb{C}})$. With some more linear algebra we can describe $\text{Rot}(\hat{\mathbb{C}})$ explicitly as a subgroup of $\text{PSL}_2(\mathbb{C})$.

If $m \in M_2(\mathbb{C})$ is a 2-by-2 complex matrix then its adjoint is

$$m^* = \overline{m^t},$$

where the overbar denotes complex conjugation, i.e.,

$$m^*_{ij} = \overline{m_{ji}} \quad \text{for } i, j = 1, 2.$$

The adjoint is characterized by the condition

$$\langle mx, y \rangle = \langle x, m^*y \rangle \quad \text{for all } x, y \in \mathbb{C}^2,$$
where now \( \langle \ , \ \rangle \) is the complex inner product
\[
\langle x, y \rangle = \sum x_i y_i.
\]
The role of the adjoint in the algebra of complex matrices is analogous to the role of the conjugate in the algebra of complex numbers. The matrix \( u \) is unitary if
\[
u^* u = I.
\]
(This condition generalizes the unit complex numbers.) Equivalently,
\[
\langle ux, uy \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{C}^2.
\]
The unitary matrices form a group \( U_2(\mathbb{C}) \). The special unitary matrices
\[
SU_2(\mathbb{C}) = \{ u \in U_2(\mathbb{C}) : \det u = 1 \}
\]
form a subgroup. A matrix is special unitary if and only if it takes the form
\[
u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad |a|^2 + |b|^2 = 1.
\]
The projective unitary group is
\[
PU_2(\mathbb{C}) = U_2(\mathbb{C})/(U_2(\mathbb{C}) \cap \mathbb{C}^* I),
\]
and the projective special unitary group is
\[
PSU_2(\mathbb{C}) = SU_2(\mathbb{C})/(SU_2(\mathbb{C}) \cap \mathbb{C}^* I) = SU_2(\mathbb{C})/\{\pm I\}.
\]
There is an isomorphism \( PU_2(\mathbb{C}) \cong PSU_2(\mathbb{C}) \), and the group \( PSU_2(\mathbb{C}) \) can be more convenient to work with since its elements are two-element cosets \( \{\pm u\} \).

**Theorem 0.2.** As a subgroup of \( PSL_2(\mathbb{C}) \), \( \text{Rot}(\hat{\mathbb{C}}) = PSU_2(\mathbb{C}) \).

Here is an elegant proof, which incidentally shows that \( \text{Rot}(\hat{\mathbb{C}}) \) is a group without reference to \( SO_3(\mathbb{R}) \). We show first that any rotation lies in \( PSU_2(\mathbb{C}) \), second that any element of \( PSU_2(\mathbb{C}) \) is a rotation.

A short calculation shows that if the antipodal pair \( p, -p \in S^2 \setminus \{n, s\} \) have stereographic images \( z, z^* \in \mathbb{C} \), then \( z^* = -1/\overline{z} \), where the overbar is complex conjugation. Now let \( f \) be a rotation of \( \hat{\mathbb{C}} \) induced by a rotation \( r \) of \( S^2 \). Let a matrix describing \( f \) be
\[
m_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(m_f) = 1.
\]
Since \( r \) takes antipodal pairs to antipodal pairs, \( f \) must satisfy the corresponding relation
\[
f(z^*) = f(z)^* \text{ for all } z \in \mathbb{C} \setminus \{0\}.
\]
This condition is that for some \( \lambda \in \mathbb{C}^* \),
\[
d = \lambda \overline{a}, \quad a = \lambda d, \quad c = -\lambda \overline{b}, \quad b = -\lambda \overline{c}.
\]
These relations and the relation \( ad - bc = 1 \) combine to show that \( \lambda = 1 \) and therefore \( m_f \in PSU_2(\mathbb{C}) \).

For the converse, let \( f \) have matrix
\[
m_f = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \in PSU_2(\mathbb{C}).
\]
If \( f(0) = 0 \) then \( f(z) = e^{i\alpha} z \) for some \( \alpha \), so \( f \) is a rotation. If \( f(0) = z \neq 0 \) then some rotation \( f_z \in \text{Rot}(\hat{\mathbb{C}}) \subset PSU_2(\mathbb{C}) \) takes \( z \) to 0, and so the composition
g = f^{-1}_{z} \circ f \in \text{PSU}_2(\mathbb{C}) \text{ fixes } 0 \text{ and is thus a rotation. Therefore } f = f_{z} \circ g \text{ is also a rotation, and the proof is complete.}

The two theorems combine to show that

\text{PSU}_2(\mathbb{C}) \cong \text{SO}_3(\mathbb{R}).

The next result says how to compute in \text{PSU}_2(\mathbb{C}) while thinking of \text{Rot}(S^2). For any rotation \(r_{p,\alpha}\) of \(S^2\), let \(f_{\pi(p),\alpha}\) denote the corresponding rotation of \(\hat{\mathbb{C}}\).

**Theorem 0.3.** Let \(p = (p_1, p_2, p_3) \in S^2\) and let \(\alpha \in \mathbb{R}\). Then

\[
f_{\pi(p),\alpha} = \begin{bmatrix}
\cos \frac{\alpha}{2} + ip_3 \sin \frac{\alpha}{2} & -p_2 \sin \frac{\alpha}{2} + ip_1 \sin \frac{\alpha}{2} & \\
p_2 \sin \frac{\alpha}{2} + ip_1 \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} - ip_3 \sin \frac{\alpha}{2} & \\
\end{bmatrix}.
\]

Here is the proof. Either by geometry or by a calculation using the commutative diagram from earlier, the rotation \(r_{n,\alpha}\) of \(S^2\) induces the automorphism \(f_{\infty,\alpha}(z) = e^{i\alpha}z\) of \(\hat{\mathbb{C}}\), i.e., under a slight abuse of notation,

\[f_{\infty,\alpha} = \begin{bmatrix}
e^{i\alpha/2} & 0 & \\
0 & e^{-i\alpha/2} & \\
\end{bmatrix}.
\]

Next consider the rotation \(r_{(0,1,0),\phi}\) of \(S^2\) counterclockwise about the positive \(x_2\)-axis through angle \(\phi\). We will find the corresponding rotation \(f_{i,\phi}\) of \(\hat{\mathbb{C}}\). A rotation \(r\) of \(S^2\) takes \((0,1,0)\) to \(n\) and \((0,-1,0)\) to \(s\); the corresponding rotation \(f\) of \(\hat{\mathbb{C}}\) takes \(i\) to \(\infty\) and \(-i\) to \(0\), so it takes the form

\[f(z) = k \frac{z + i}{z - i}\]

for some nonzero constant \(k\). Since \(r_{(0,1,0),\phi} = r^{-1} \circ r_{n,\phi} \circ r\), the corresponding result in \(\text{Rot}(\hat{\mathbb{C}})\) is

\[f_{i,\phi} = f^{-1} \circ f_{\infty,\phi} \circ f,
\]
or

\[f \circ f_{i,\phi} = f_{\infty,\phi} \circ f.
\]

Thus for all \(z \in \hat{\mathbb{C}}\),

\[k \frac{f_{i,\phi}(z) + i}{f_{i,\phi}(z) - i} = e^{i\phi}k \cdot \frac{z + i}{z - i}\]

The \(k\) cancels, leaving

\[e^{-i\phi/2}(f_{i,\phi}(z) + i)(z - i) = e^{i\phi/2}(f_{i,\phi}(z) - i)(z + i),\]

and some algebra gives

\[f_{i,\phi} = \begin{bmatrix}
\cos \frac{\phi}{2} & -\sin \frac{\phi}{2} & \\
\sin \frac{\phi}{2} & \cos \frac{\phi}{2} & \\
\end{bmatrix}.
\]

Now let the point \(p \in S^2\) have spherical coordinates \((1, \theta, \phi)\), meaning that

\[
\cos \theta = p_1/\sqrt{p_1^2 + p_2^2}, \quad \sin \theta = p_2/\sqrt{p_1^2 + p_2^2}, \quad \cos \phi = p_3, \quad \sin \phi = \sqrt{p_1^2 + p_2^2}.
\]

(See figure 2.) To carry out \(r_{p,\alpha}\), move \(p\) to the north pole via rotations about the north pole and \((0,1,0)\), rotate about the north pole by \(\alpha\), and restore \(p\); to wit,

\[r_{p,\alpha} = r_{n,\theta} \circ r_{(0,1,0),\phi} \circ r_{n,\alpha} \circ r_{(0,1,0),-\phi} \circ r_{n,-\theta}.\]
The corresponding rotation of $\hat{C}$ is

$$f_{\pi(p),\alpha} = \begin{bmatrix} e^{i\theta/2} & 0 & e^{-i\theta/2} \\ 0 & e^{i\alpha/2} & 0 \\ e^{-i\theta/2} & e^{-i\alpha/2} & 0 \end{bmatrix} \begin{bmatrix} \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\ \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} & 0 & e^{i\theta/2} \\ 0 & e^{i\alpha/2} & 0 \\ e^{-i\alpha/2} & 0 & 1 \end{bmatrix}.$$

Multiplying this out and using a little trigonometry gives the result.

**Figure 2.** Spherical coordinates