THE RIEMANN MAPPING THEOREM

Theorem 0.1 (Riemann Mapping Theorem)**.** *Let Ω be a a simply connected region in $\mathbb{C}$ that is not all of $\mathbb{C}$. Let D be the unit disk. Then there exists an analytic bijection*

$$f : \Omega \xrightarrow{\sim} D.$$

For each point $z_0 \in \Omega$, there is a unique such map f such that

$$f(z_0) = 0, \quad f'(z_0) \in \mathbb{R}^+.$$

The proof of uniqueness, granting existence, is easy. Suppose that two maps

$$f_1, f_2 : \Omega \xrightarrow{\sim} D$$

satisfy the normalizing conditions, $f_1(z_0) = f_2(z_0) = 0$ and $f_1'(z_0), f_2'(z_0) \in \mathbb{R}^+$. Consider the map

$$f_2 \circ f_1^{-1} : D \xrightarrow{\sim} D.$$

This automorphism of D fixes 0, and so it is a rotation. That is,

$$f_2(z) = e^{i\theta} f_1(z) \quad \text{for some } \theta,$$

and so

$$f_2'(z_0) = e^{i\theta} f_1'(z_0).$$

But these are both real and positive, forcing $e^{i\theta} = 1$, i.e., $f_2 = f_1$.

The proof of existence breaks nicely into parts that use different ideas. Consider a family of functions,

$$\mathcal{F} = \{\text{analytic, injective } f : \Omega \longrightarrow D \text{ such that } f(z_0) = 0\}.$$

The argument will show that

(A) $\mathcal{F}$ is nonempty.

(B) If some $f \in \mathcal{F}$ satisfies

$$|f'(z_0)| \geq |g'(z_0)| \quad \text{for all } g \in \mathcal{F}$$

then f is surjective.

(C) $\mathcal{F}$ is equicontinuous. So the Arzela–Ascoli Theorem and some other general results complete the argument.

(A) To show that $\mathcal{F}$ is nonempty amounts to finding a suitable map from Ω to D.

Some complex number a does not belong to Ω since Ω is not all of $\mathbb{C}$, and after a translation we may assume that $a = 0$. There is a path γ from 0 to ∞ in the complement of Ω since Ω is simply connected. So we can define an analytic square root

$$r : \Omega \longrightarrow \mathbb{C}, \quad r(z) = \sqrt{z}.$$

Note that r can not assume both some value w and its opposite $-w$, because

$$r(z) = w \implies z = w^2 \quad \text{and} \quad r(z') = -w \implies z' = (-w)^2 = w^2.$$

That is, the only candidate input z' to be taken by r to $-w$ is the input z taken to w instead. Now let w_0 be any value taken by r. By the Open Mapping Theorem, $r(\Omega)$ contains some disk $N(w_0, \varepsilon)$ about w_0, and therefore $r(\Omega)$ is disjoint from the opposite disk,

$$r(\Omega) \cap N(-w_0, \varepsilon) = \emptyset.$$

Follow r by a translation and a scale to get a map $g : \Omega \longrightarrow \mathbb{C}$ such that

$$g(\Omega) \cap D = \emptyset.$$

Consequently,

$$(1/g)(\Omega) \subset D.$$

Let $p = (1/g)(z_0)$, and let $f = T_p \circ (1/g)$, where T_p is the usual automorphism of D that takes p to 0,

$$T_p z = \frac{z - p}{1 - \overline{p}z}.$$

Then f is an element of $\mathcal{F}$.

(B) We need to show that if some $f \in \mathcal{F}$ has maximal absolute derivative at z_0 then f surjects. We will show the contrapositive, that if f does not surject then some $g \in \mathcal{F}$ has larger absolute derivative at z_0.

So, suppose that f does not surject. Then f misses some point $w \in D$. Define

$$g = T_{w'} \circ \mathrm{sqrt} \circ T_w \circ f,$$

where we are taking some well defined branch of square root on the simply connected set $(T_w \circ f)(\Omega)$, and $w' = \sqrt{-w}$. It follows that

$$f = T_w^{-1} \circ \mathrm{sq} \circ T_{w'}^{-1} \circ g.$$

The self-map of the disk

$$s = T_w^{-1} \circ \mathrm{sq} \circ T_{w'}^{-1} : D \longrightarrow D$$

fixes 0, and because of the square, it is not an automorphism. Therefore,

$$|s'(0)| < 1,$$

and so since $f = s \circ g$ and $g(z_0) = 0$ the chain rule gives

$$|f'(z_0)| = |s'(0)|\,|g'(z_0)| < |g'(z_0)|.$$

This completes the argument.

(C) Next we show that $\mathcal{F}$ is equicontinuous. So let $\varepsilon > 0$ be given. We may assume that $\varepsilon < 1$.

Consider any point z of Ω. Since Ω is open, some closed disk $\overline{N(z, 2\rho)}$ lies in Ω. Let γ denote the circle of radius 2ρ about z,

$$\gamma = \{\zeta : |\zeta - z| = 2\rho\}.$$

Then for any $\tilde{z}$ such that $|\tilde{z} - z| < \rho\varepsilon$, and for any $f \in \mathcal{F}$,

$$\begin{aligned}
|f(\tilde{z}) - f(z)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - \tilde{z}} - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \left(\frac{1}{\zeta - \tilde{z}} - \frac{1}{\zeta - z} \right) f(\zeta)\,\mathrm{d}\zeta \right| \\
&= \frac{1}{2\pi} \left| \int_\gamma \frac{(\tilde{z} - z) f(\zeta)\,\mathrm{d}\zeta}{(\zeta - \tilde{z})(\zeta - z)} \right| \\
&< \frac{1}{2\pi} \frac{\rho\varepsilon}{\rho \cdot 2\rho} \int_\gamma |\mathrm{d}\zeta| \\
&= \frac{1}{2\pi} \frac{\rho\varepsilon}{\rho \cdot 2\rho} 2\pi \cdot 2\rho \\
&= \varepsilon.
\end{aligned}$$

This shows that the definition of continuity is satisfied at z by $\delta = \rho\varepsilon$.

Recall a theorem due to Weierstrass that we used earlier to show that power series are analytic:

Theorem 0.2 (Weierstrass)**.** *Let Ω be a region in $\mathbb{C}$. Consider a sequence of analytic functions on Ω,*

$$\{f_0, f_1, f_2, \dots\} : \Omega \longrightarrow \mathbb{C}.$$

Suppose that the sequence converges on Ω to a limit function

$$f : \Omega \longrightarrow \mathbb{C}$$

and that the convergence is uniform on compact subsets of Ω. Then

(1) *The limit function f is analytic.*
(2) *The sequence $\{f_n'\}$ of derivatives converges on Ω to the derivative f' of the limit function.*
(3) *This convergence is also uniform on compact subsets of Ω.*

Recall also the Hurwitz Theorem:

Theorem 0.3 (Hurwitz)**.** *If the f_n are injective, and f is not constant, then f is injective.*

To complete the proof of the Riemann Mapping Theorem, let $\{f_n\}$ be a sequence from $\mathcal{F}$ such that

$$\lim_n \{|f_n'(z_0)|\} = \sup_{f \in \mathcal{F}} \{|f'(z_0)|\}.$$

(The supremum is readily seen to be finite by Cauchy's estimate, but we don't even need this.) By the Arzela–Ascoli Theorem, a subsequence converges to some function $f : \Omega \longrightarrow D$, and the convergence is uniform on compact subsets. The Weierstrass Theorem says that f is analytic and $|f'(z_0)|$ is maximal. The Hurwitz Theorem says that f is injective. Part B says that f is surjective. This completes the proof.