**The Riemann Mapping Theorem**

**Theorem 0.1** (Riemann Mapping Theorem). Let $\Omega$ be a simply connected region in $\mathbb{C}$ that is not all of $\mathbb{C}$. Let $D$ be the unit disk. Then there exists an analytic bijection $f : \Omega \sim \rightarrow D$.

For each point $z_0 \in \Omega$, there is a unique such map $f$ such that $f(z_0) = 0$, $f'(z_0) \in \mathbb{R}^+$. The proof of uniqueness, granting existence, is easy. Suppose that two maps $f_1, f_2 : \Omega \sim \rightarrow D$ satisfy the normalizing conditions, $f_1(z_0) = f_2(z_0) = 0$ and $f_1'(z_0), f_2'(z_0) \in \mathbb{R}^+$. Consider the map $f_2 \circ f_1^{-1} : D \sim \rightarrow D$.

This automorphism of $D$ fixes 0, and so it is a rotation. That is, $f_2(z) = e^{i\theta}f_1(z)$ for some $\theta$, and so $f_2'(z_0) = e^{i\theta}f_1'(z_0)$.

But these are both real and positive, forcing $e^{i\theta} = 1$, i.e., $f_2 = f_1$.

The proof of existence breaks nicely into parts that use different ideas. Consider a family of functions, $\mathcal{F} = \{\text{analytic, injective } f : \Omega \rightarrow D \text{ such that } f(z_0) = 0\}$.

The argument will show that

(A) $\mathcal{F}$ is nonempty.
(B) If some $f \in \mathcal{F}$ satisfies

$$|f'(z_0)| \geq |g'(z_0)|$$

for all $g \in \mathcal{F}$

then $f$ is surjective.
(C) $\mathcal{F}$ is equicontinuous. So the Arzela–Ascoli Theorem and some other general results complete the argument.

(A) To show that $\mathcal{F}$ is nonempty amounts to finding a suitable map from $\Omega$ to $D$.

Some complex number $a$ does not belong to $\Omega$ since $\Omega$ is not all of $\mathbb{C}$, and after a translation we may assume that $a = 0$. There is a path $\gamma$ from 0 to $\infty$ in the complement of $\Omega$ since $\Omega$ is simply connected. So we can define an analytic square root

$$r : \Omega \rightarrow \mathbb{C}, \quad r(z) = \sqrt{z}.$$ 

Note that $r$ cannot assume both some value $w$ and its opposite $-w$, because $r(z) = w \implies z = w^2$ and $r(z') = -w \implies z' = (-w)^2 = w^2$. 

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That is, the only candidate input \( z' \) to be taken by \( r \) to \(-w\) is the input \( z \) taken to \( w \) instead. Now let \( w_0 \) be any value taken by \( r \). By the Open Mapping Theorem, \( r(\Omega) \) contains some disk \( N(w_0, \varepsilon) \) about \( w_0 \), and therefore \( r(\Omega) \) is disjoint from the opposite disk,

\[
r(\Omega) \cap N(-w_0, \varepsilon) = \emptyset.
\]

Follow \( r \) by a translation and a scale to get a map \( g : \Omega \rightarrow \mathbb{C} \) such that

\[
g(\Omega) \cap D = \emptyset.
\]

Consequently,

\[
(1/g)(\Omega) \subset D.
\]

Let \( p = (1/g)(z_0) \), and let \( f = T_p \circ (1/g) \), where \( T_p \) is the usual automorphism of \( D \) that takes \( p \) to 0,

\[
T_p z = \frac{z - p}{1 - \overline{p} z}.
\]

Then \( f \) is an element of \( \mathcal{F} \).

(B) We need to show that if some \( f \in \mathcal{F} \) has maximal absolute derivative at \( z_0 \) then \( f \) surjects. We will show the contrapositive, that if \( f \) does not surject then some \( g \in \mathcal{F} \) has larger absolute derivative at \( z_0 \).

So, suppose that \( f \) does not surject. Then \( f \) misses some point \( w \in D \). Define

\[
g = T_{w'} \circ \text{sqrt} \circ T_w \circ f,
\]

where we are taking some well defined branch of square root on the simply connected set \((T_w \circ f)(\Omega)\), and \( w' = \sqrt{-w} \). It follows that

\[
f = T_w^{-1} \circ \text{sq} \circ T_{w'}^{-1} \circ g.
\]

The self-map of the disk

\[
s = T_w^{-1} \circ \text{sq} \circ T_{w'}^{-1} : D \rightarrow D
\]

fixes 0, and because of the square, it is not an automorphism. Therefore,

\[
|s'(0)| < 1,
\]

and so since \( f = s \circ g \) and \( g(z_0) = 0 \) the chain rule gives

\[
|f'(z_0)| = |s'(0)||g'(z_0)| < |g'(z_0)|.
\]

This completes the argument.

(C) Next we show that \( \mathcal{F} \) is equicontinuous. So let \( \varepsilon > 0 \) be given. We may assume that \( \varepsilon < 1 \).

Consider any point \( z \) of \( \Omega \). Since \( \Omega \) is open, some closed disk \( \overline{N(z, 2\rho)} \) lies in \( \Omega \). Let \( \gamma \) denote the circle of radius \( 2\rho \) about \( z \),

\[
\gamma = \{ \zeta : |\zeta - z| = 2\rho \}.
\]
Then for any $\tilde{z}$ such that $|\tilde{z} - z| < \rho \varepsilon$, and for any $f \in F$,

$$|f(\tilde{z}) - f(z)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - \tilde{z}} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} \right|$$

$$= \frac{1}{2\pi} \left| \int_{\gamma} \left( \frac{1}{\zeta - \tilde{z}} - \frac{1}{\zeta - z} \right) f(\zeta) d\zeta \right|$$

$$= \frac{1}{2\pi} \left| \int_{\gamma} \frac{(\tilde{z} - z) f(\zeta) d\zeta}{(\zeta - \tilde{z})(\zeta - z)} \right|$$

$$< \frac{1}{2\pi} \frac{\rho \varepsilon}{\rho \cdot 2\rho} \int_{\gamma} |d\zeta|$$

$$= \frac{1}{2\pi} \frac{\rho \varepsilon}{\rho \cdot 2\rho} 2\pi \cdot 2\rho$$

$$= \varepsilon.$$

This shows that the definition of equicontinuity is satisfied at $z$ by $\delta = \rho \varepsilon$.

Recall a theorem due to Weierstrass that we used earlier to show that power series are analytic:

**Theorem 0.2 (Weierstrass).** Let $\Omega$ be a region in $\mathbb{C}$. Consider a sequence of analytic functions on $\Omega$,

$$\{f_0, f_1, f_2, \ldots \} : \Omega \rightarrow \mathbb{C}.$$

Suppose that the sequence converges on $\Omega$ to a limit function

$$f : \Omega \rightarrow \mathbb{C}$$

and that the convergence is uniform on compact subsets of $\Omega$. Then

1. The limit function $f$ is analytic.
2. The sequence $\{f'_n\}$ of derivatives converges on $\Omega$ to the derivative $f'$ of the limit function.
3. This convergence is also uniform on compact subsets of $\Omega$.

Recall also the Hurwitz Theorem:

**Theorem 0.3 (Hurwitz).** If the $f_n$ are injective, and $f$ is not constant, then $f$ is injective.

To complete the proof of the Riemann Mapping Theorem, let $\{f_n\}$ be a sequence from $\mathcal{F}$ such that

$$\lim_n |f'_n(z_0)| = \sup_{f \in \mathcal{F}} |f'(z_0)|.$$

(The supremum is readily seen to be finite by Cauchy’s estimate, but we don’t even need this.) By the Arzela–Ascoli Theorem, a subsequence converges to some function $f : \Omega \rightarrow D$, and the convergence is uniform on compact subsets. The Weierstrass Theorem says that $f$ is analytic and $|f'(z_0)|$ is maximal. The Hurwitz Theorem says that $f$ is injective. Part B says that $f$ is surjective. This completes the proof.