THE RESIDUE THEOREM AND ITS CONSEQUENCES

1. Introduction

With Laurent series and the classification of singularities in hand, it is easy to prove the Residue Theorem. In addition to being a handy tool for evaluating integrals, the Residue Theorem has many theoretical consequences. This writeup presents the Argument Principle, Rouché's Theorem, the Local Mapping Theorem, the Open Mapping Theorem, the Hurwitz Theorem, the general Casorati–Weierstrass Theorem, and Riemann's Theorem.

2. The Residue Theorem

Definition 2.1. *Let c be a point in $\mathbb{C}$, and let f be a function that is meromorphic at c. Let the Laurent series of f about c be*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n,$$

where $a_n = 0$ for all n less than some N. Then the **residue** *of f at c is*

$$\operatorname{Res}_c(f) = a_{-1}.$$

Theorem 2.2 (Residue Theorem). *Let Ω be a region and let f be meromorphic on Ω. Let γ be a simple closed contractible counterclockwise curve in Ω, and suppose that f is analytic on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \sum_{c \text{ inside } \gamma} \operatorname{Res}_c(f).$$

Proof. Although the sum in the residue theorem is taken over an uncountable set, it has only finitely many nonzero terms, those arising from the points c inside γ where f has poles. The Deformation Theorem lets us replace γ by finitely many small counterclockwise loops, one around each such c. The basic result that

$$\frac{1}{2\pi i}\int_{|\zeta-c|=\varepsilon} (\zeta-c)^n\,\mathrm{d}\zeta = \begin{cases} 1 & \text{if } n=-1, \\ 0 & \text{otherwise.} \end{cases}$$

combines with the Laurent series representation to show that the integral over each such loop is the relevant residue. □

3. The Argument Principle

Theorem 3.1 (Argument Principle). *Let γ be a simple closed counterclockwise curve. Let f be analytic and nonzero on γ and meromorphic inside γ. Let $Z(f)$ denote the number of zeros of f inside γ, each counted as many times as its multiplicity, and let $P(f)$ denote the number of poles of f inside γ, each counted as many times as its multiplicity. Then*

$$Z(f) - P(f) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}f}{f}.$$

Proof. The idea is that for any point c inside γ,

$$\mathrm{Res}_c(f'/f) = \mathrm{ord}_c(f),$$

and so the Argument Principle follows immediately from the Residue Theorem. Indeed, let c be any point inside γ, and let $N = \mathrm{ord}_c(f)$. The Laurent expansion of f at c is

$$f(z) = (z-c)^N \sum_{n=0}^{\infty} b_n (z-c)^n,$$

where $b_n = a_{n+N}$ and in particular $b_0 \neq 0$. Thus

$$f(z) = (z-c)^N g(z), \quad g(z) \text{ analytic and nonzero at } c.$$

Compute the logarithmic derivative of f,

$$\frac{f'(z)}{f(z)} = \frac{N(z-c)^{N-1}g(z) + (z-c)^N g'(z)}{(z-c)^N g(z)} = \frac{N}{z-c} + \frac{g'(z)}{g(z)}\,.$$

Note that $g'(z)/g(z)$ is analytic at c since the denominator doesn't vanish there. So the expression in parentheses has a simple pole at c whose residue is precisely N. The anticipated result that $\mathrm{Res}_c(f'/f) = \mathrm{ord}_c(f)$ follows. Next, note that the logarithmic derivative f'/f is analytic on γ and meromorphic inside γ. So by the residue theorem and the previous formula,

$$\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}f}{f} = \sum_{c \text{ inside } \gamma} \mathrm{Res}_c(f'/f) = \sum_{c \text{ inside } \gamma} \mathrm{ord}_c(f).$$

The last sum is $Z(f) - P(f)$, and so the proof is complete. □

For one quick application of the Argument Principle, let f be a nonvanishing entire function. We show that f has a logarithm. Consider a second entire function, defined as the sum of an integral and a constant,

$$g(z) = \int_0^z \frac{f'(\zeta)\,\mathrm{d}\zeta}{f(\zeta)} + \log(f(0)), \qquad \text{(using any value of } \log(f(0))\text{)}.$$

This function g is well defined in consequence of the Argument Principle, and its derivative is $g'(z) = f'(z)/f(z)$. To see that g is a logarithm of f, compute

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big(e^{-g(z)}f(z)\Big) = e^{-g(z)}\big(f'(z) - g'(z)f(z)\big) = 0,$$

so that $f(z) = c\,e^{g(z)}$, and then set $z = 0$ to see that $c = 1$.

4. The Winding Number

Definition 4.1. *Let $\gamma : [0,1] \longrightarrow \mathbb{C}$ be any closed rectifiable path. By the usual abuse of notation, let γ also denote the corresponding subset of $\mathbb{C}$. Consider a complex-valued function on the complement of the path,*

$$\mathrm{Ind}(\gamma,\cdot) : \mathbb{C} - \gamma \longrightarrow \mathbb{C}, \qquad \mathrm{Ind}(\gamma, z) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z}\,.$$

For any $z \in \mathbb{C} - \gamma$, the function $\mathrm{Ind}(\gamma, z)$ *is the* winding number of γ about z.

For instance, if γ is a circle traversed once counterclockwise about z then its winding number about z is 1.

Proposition 4.2. *The winding number is an integer, and it is constant on the connected components of* $\mathbb{C} - \gamma$.

This proposition is topological, and a proof for rectifiable curves is not trivial. We prove it only for piecewise $\mathcal{C}^1$-curves.

Proof. First assume that γ is $\mathcal{C}^1$. To show that the winding number is an integer, consider for any $z \in \mathbb{C} - \gamma$ the function

$$\varphi : [0,1] \longrightarrow \mathbb{C}, \qquad \varphi(t) = \int_0^t \frac{\gamma'(\tau)\,\mathrm{d}\tau}{\gamma(\tau) - z}\,.$$

By the fundamental theorem of calculus, at every point t where γ is $\mathcal{C}^1$,

$$\varphi'(t) = \frac{\gamma'(t)}{\gamma(t) - z}\,.$$

And so, using the product rule for derivatives, at every point t where γ is $\mathcal{C}^1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\varphi(t)}(\gamma(t) - z)\right) = 0.$$

Thus on each subinterval of $[0,1]$ where γ is $\mathcal{C}^1$, we have $\gamma(t) - z = c\,e^{\varphi(t)}$ for some constant c, and this constant is nonzero because z doesn't lie on γ. Further, because γ is continuous, the constants c for consecutive subintervals of $[0,1]$ where γ' is continuous must agree. Thus $\gamma(t) - z = c\,e^{\varphi(t)}$ for a single nonzero constant c for all $t \in [0,1]$. Because $\gamma(1) = \gamma(0)$, it follows that $e^{\varphi(1)} = e^{\varphi(0)} = e^0$, and so $\varphi(1) = 2\pi i n$ for some integer n. That is, $\mathrm{Ind}(\gamma, z)$ is an integer as desired.

Also, $\mathrm{Ind}(\gamma, z)$ is continuous as a function of z. Indeed, by familiar arguments, γ is disjoint from some ball B about z, and for any sequence $\{z_n\}$ in B converging to z, the functions $\{1/(\zeta - z_n)\}$ converge uniformly to $1/(\zeta - z)$ on γ, giving

$$\lim_{n\to\infty} \mathrm{Ind}(\gamma, z_n) = \frac{1}{2\pi i} \lim_{n\to\infty} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z_n} = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z} = \mathrm{Ind}(\gamma, z).$$

And so, being a continuous, integer-valued function, the winding number is constant on the connected components of its domain $\mathbb{C} - \gamma$. This completes the proof. □

5. The Argument Principle Again

With the winding number in hand, we can rephrase the Argument Principle in a way that explains its name.

Theorem 5.1 (Argument Principle, second version)**.** *Let γ be a simple closed counterclockwise curve. Let f be analytic and nonzero on γ and meromorphic inside γ. Let $Z(f)$ denote the number of zeros of f inside γ, each counted as many times as its multiplicity, and let $P(f)$ denote the number of poles of f inside γ, each counted as many times as its multiplicity. Then*

$$Z(f) - P(f) = \mathrm{Ind}(f \circ \gamma, 0).$$

That is, the theorem is called the Argument Principle because the number of zeros minus poles of f inside γ is the number of times that the argument of $f \circ \gamma$ increases by 2π.

For example, consider the polynomial

$$f(z) = z^4 - 8z^3 + 3z^2 + 8z + 3.$$

To count the roots of f in the right half plane, let D be a large disk centered at the origin, large enough to contain all the roots of f. Let γ be the boundary of the right half of D. Thus γ is the union of a segment of the imaginary axis and a semicircle. The values of f on the imaginary axis are

$$f(iy) = (y^4 - 3y^2 + 3) + i(8y^3 + 8y).$$

The real part of $f(iy)$ is always positive, and so f takes the imaginary axis into the right half plane. On the semicircle, $f(z)$ behaves qualitatively as z^4. Therefore the path $\Gamma = f \circ \gamma$ winds twice around the origin, showing that f has two roots in the right half plane.

6. Rouché's Theorem

Theorem 6.1 (Rouché's Theorem)**.** *Let γ be a simple closed counterclockwise curve. Let f and g be analytic on and inside γ, and let them satisfy the condition*

$$|f - g| < |g| \quad \text{on } \gamma.$$

Then f and g have the same number of roots inside γ.

The usual application is that f is some given function and g is its dominant term on γ, easier to analyze than the more complicated f.

Proof. The given condition shows that f and g are nonzero on γ, making the quotient f/g nonzero and analytic on γ. By the Argument Principle,

$$Z(f) - Z(g) = Z(f/g) - P(f/g) = \operatorname{Ind}((f/g) \circ \gamma, 0).$$

Note that on γ,

$$\left| \frac{f}{g} - 1 \right| = \left| \frac{f - g}{g} \right| < 1.$$

So $(f/g) \circ \gamma$ does not wind about the origin, i.e., $\operatorname{Ind}((f/g) \circ \gamma, 0) = 0$. This completes the proof. ☐

For example, consider the polynomial

$$f(z) = z^7 - 2z^5 + 6z^3 - z + 1.$$

To count the roots of f in the unit disk $D_1 = \{|z| < 1\}$, let $g(z)$ be the dominant term of f on its boundary, the unit circle,

$$g(z) = 6z^3.$$

Then on the unit circle we have

$$|f(z) - g(z)| = |z^7 - 2z^5 - z + 1| \leq 5 < 6 = |6z^3| = |g(z)|.$$

Since g has three roots in D_1, so does f. To count the roots of f in the disk $D_2 = \{|z| < 2\}$, let $g(z)$ be the dominant term of f on the circle of radius 2,

$$g(z) = z^7.$$

On the boundary circle of D_2 we have

$$|f(z) - g(z)| = |-2z^5 + 6z^3 - z + 1| \leq 115 < 128 = |z^7| = |g(z)|.$$

Since g has seven roots in D_2, so does f.

7. Local Analysis of Analytic Functions

Theorem 7.1 (Local Mapping Theorem)**.** *Suppose f is analytic at z_0 and that $f(z) - w_0$ has a zero of order n at z_0. Then f is n-to-1 near z_0. More specifically, for any sufficiently small ball $B(z_0, \varepsilon)$ about z_0 there is corresponding ball $B(w_0, \delta)$ about w_0 such that for all $w \in B(w_0, \delta) - \{w_0\}$, the equation*

$$f(z) = w$$

has n distinct roots in $B(z_0, \varepsilon)$.

Proof. Since f is not identically w_0, its w_0-points are isolated. So on some closed ball $\overline{B} = \overline{B(z_0, \varepsilon)}$, f takes the value w_0 only at z_0. By further shrinking ε if necessary, we may assume also that the only possible zero of f' in $\overline{B}$ is at z_0. Let γ be the boundary circle of $\overline{B}$. Then $f \neq w_0$ on γ, and by the Argument Principle,

$$n = \operatorname{Ind}(f \circ \gamma, w_0).$$

Since the winding number is constant on the connected components of $\mathbb{C} - f \circ \gamma$, it follows that for all w close enough to w_0,

$$\operatorname{Ind}(f \circ \gamma, w) = \operatorname{Ind}(f \circ \gamma, w_0) = n.$$

But $\operatorname{Ind}(f \circ \gamma, w)$ is the number of z-values in B such that $f(z) = w$, counting multiplicity, and f doesn't take any value with multiplicity greater than one, except possibly w_0. This completes the proof. □

Corollary 7.2 (Open Mapping Theorem)**.** *Let f be analytic and nonconstant. Then f maps open sets to open sets, and at any point z_0 such that $f'(z_0) \neq 0$, f is a local homeomorphism.*

Proof. To show that f is an open mapping, let S be an open set in its domain. Consider any point $w_0 \in f(S)$, i.e., $w_0 = f(z_0)$ for some $z_0 \in S$. For all small ε, the ball $B(z_0, \varepsilon)$ lies in S. The Local Mapping Theorem says that for all sufficiently small ε, some ball $B(w_0, \delta)$ lies in $f(B(z_0, \varepsilon))$, a subset of $f(S)$. Thus $f(S)$ is open.

To show that f is a local homeomorphism if $f'(z_0) \neq 0$, note that this condition means that $n = 1$ in the Local Mapping Theorem. Thus f is a bijection between $B(w_0, \delta)$ and $f^{-1}(B(w_0, \delta))$, and the previous paragraph has shown that f^{-1} is continuous. □

Note how the Local Mapping Theorem reproves the Maximum Principle more convincingly: an analytic function f maps open sets to open sets, and so its modulus $|f|$ can't take a maximum on an open set.

To understand the local mapping more explicitly, note that near z_0 we have

$$f(z) - w_0 = (z - z_0)^n g(z), \quad g(z_0) \neq 0.$$

By continuity, $|g(z) - g(z_0)| < |g(z_0)|$ for all z near z_0. So we can take a branch of the nth root of g near z_0. Call it h. Thus

$$w - w_0 = f(z) - w_0 = ((z - z_0)h(z))^n = \zeta^n, \quad \zeta = (z - z_0)h(z),$$

and the map $z \mapsto \zeta$ is a homeomorphism near z_0. Thus the general map f is locally a translation of a homeomorphism, followed by an nth power.

In fact, when the local inverse f^{-1} of an analytic function exists, it is also analytic. The point is that by the Local Mapping Theorem, at each point z where

f is locally invertible, necessarily $f'(z) \neq 0$. And then a calculation from one-variable calculus, equally valid over the complex numbers as over the real numbers, shows that the derivative of the inverse of f at $f(z)$ exists and equals the reciprocal of the derivative of f at z. A situation like the real function $f(x) = x^3$, which is differentiable and invertible but whose inverse is not differentiable at 0, cannot arise.

8. The Hurwitz Theorem

Theorem 8.1 (The Hurwitz Theorem)**.** *Let*

$$\{f_n\} : \Omega \longrightarrow \mathbb{C}$$

be a sequence of analytic functions that never vanish on Ω, and suppose that the sequence converges uniformly on compact subsets of Ω to a limit function

$$f : \Omega \longrightarrow \mathbb{C}.$$

Then either f never vanishes on Ω as well, or f is identically zero.

Proof. We know that f is analytic. Assume that it is not identically zero. Then any zeros that it may have must be isolated. So given any point z of Ω, some small circle γ about z contains no zeros of f except possibly z itself. But the sequence $\{f_n'/f_n\}$ converges to f'/f uniformly on γ (exercise), and so the Argument Principle shows that the number of zeros of f inside γ is

$$Z(f) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}f}{f} = \lim_{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}f_n}{f_n} = \lim_{n\to\infty} Z(f_n) = 0.$$

So in particular, $f(z) \neq 0$. Since z is an arbitrary point of Ω, the proof is complete. □

Corollary 8.2. *Let*

$$\{f_n\} : \Omega \longrightarrow \mathbb{C}$$

be a sequence of analytic functions that are injective on Ω, and suppose that the sequence converges uniformly on compact subsets of Ω to a limit function

$$f : \Omega \longrightarrow \mathbb{C}.$$

Then either f is injective on Ω as well, or f is constant.

Proof. Suppose that $f(z_1) = f(z_2)$ for distinct points $z_1, z_2 \in \Omega$. Some open ball B about z_1 lies in Ω and misses z_2. The functions $g_n : B \longrightarrow \mathbb{C}$ given by $g_n(z) = f_n(z) - f_n(z_2)$ are never 0, while their limit function $g : B \longrightarrow \mathbb{C}$ given by $g(z) = f(z) - f(z_2)$ is 0 at $z = z_1$. By the theorem, g is identically 0 on B. That is, f is the constant function $f(z_2)$ on B, and so f is constant on Ω by the uniqueness theorem. □

9. Behavior at infinity

If f is analytic on $\mathbb{C}$ except for finitely many singularities then f is analytic on $\{|z| > r\}$ for some positive r. Define a function g on inputs near 0 that behaves as f behaves for large inputs,

$$g : \{\zeta : 0 < |\zeta| < 1/r\} \longrightarrow \mathbb{C}, \qquad g(\zeta) = f(1/\zeta).$$

Then g has an isolated singularity at 0. The nature of the singularity of f at ∞ is *defined* to be the nature of the singularity of g at 0. For example, a polynomial f of degree n has a pole of order n at ∞, because in this case

$$g(\zeta) = f(1/\zeta) = \sum_{j=0}^{n} a_j \zeta^{-j}, \quad a_n \neq 0.$$

A similar calculation shows that a rational function

$$f(z) = \frac{p(z)}{q(z)}, \qquad \deg(p) = n,\ \deg(q) = m$$

has order $m - n$ at infinity, giving the pleasant relation

$$\sum_{c \in \mathbb{C} \cup \infty} \mathrm{ord}_c(f) = 0.$$

An entire transcendental function f has an essential singularity at ∞ since

$$g(\zeta) = f(1/\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{-n}.$$

And the principal part of any Laurent expansion has a removable singularity at ∞. (This finishes exercise 5(a); in 5(b) the singularity at ∞ is not isolated since $g(\zeta) = f(1/\zeta)$ isn't analytic in any punctured disk about 0.)

What's going on here is that the Riemann sphere, while globally distinct from $\mathbb{C}$, is indistinguishable from $\mathbb{C}$ in the small. To study a function at ∞, we use the mapping

$$z \longmapsto 1/z \overset{\text{call}}{=} \zeta$$

to take a neighborhood of ∞ homeomorphically to a neighborhood of 0, since we understand how to analyze singularities at 0. The notion we are tiptoeing around here is that of a *manifold*, loosely a topological space that is locally Euclidean. More generally than studying functions at infinity, if f has a nonisolated singularity at c and some mapping φ takes a neighborhood of a point $p \in \mathbb{C}$ analytically and homeomorphically to a neighborhood of c, then the singularity of f at c is of the same type as the singularity of $f \circ \varphi$ at p. Proving this is an exercise in manipulating Laurent series. (Changing variables like this makes exercise 5(c) easy at $z = 1/n$ where n is a nonzero integer. Let $z = 1/(\zeta + n)$, so that $\zeta = 1/z - n$, in order to study $\sin(\pi/z)$ at $1/n$ by studying $\sin \pi(\zeta + n\pi)$ at 0. The singularity at 0 is nonisolated.)

10. Function-Theoretic results

Using the ideas here makes it easy to generalize the Casorati-Weierstrass theorem:

Theorem 10.1 (Casorati–Weierstrass Theorem, version 2)**.** *If f has an essential singularity at ∞ then for all large enough values R, the set*

$$\{f(z) : |z| > R\}$$

is dense in $\mathbb{C}$.

Proof. To say that f has an essential singularity at ∞ is to say that it has a two-sided expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{for large } z,$$

with the principal part $\sum_{n=-\infty}^{-1} a_n z^n$ convergent for large z and extending continuously to 0 at $z = \infty$, and with $\sum_{n=0}^{\infty} a_n z^n$ an entire transcendental function. The principal part has absolute value less than $\varepsilon/2$ for z large enough, while the entire transcendental function gets within $\varepsilon/2$ of any $c \in \mathbb{C}$ for infinitely many large z by the previous version of Casorati–Weierstrass. The result follows. □

In fact, once we think in terms of manifolds, there is nothing special about infinity. The final Casorati-Weierstrass theorem is

Theorem 10.2 (Casorati–Weierstras Theorem version 3)**.** *If f has an essential singularity at a point $c \in \mathbb{C} \cup \infty$ then for any small enough neighborhood N of c, the set*

$$f(N - \{c\})$$

is dense in $\mathbb{C}$.

Proof. The result is already established if $c = \infty$. If $c \in \mathbb{C}$ instead then let $g(z) = f(z + c)$, which has an essential singularity at 0, and then let $h(z) = g(1/z)$, which has an essential singularity at ∞. Since h takes large inputs to a dense set of outputs, g takes inputs near 0 to a dense set of outputs, and so f takes inputs near c to a dense set of outputs. □

The summary theorem about singularities is called

Theorem 10.3 (Riemann's Theorem)**.** *Let f have an isolated singularity at the point $c \in \mathbb{C} \cup \infty$. The singularity is*

- *removable if and only if f is bounded near c,*
- *a pole if and only if $|f(z)| \to +\infty$ as $z \to c$,*
- *essential if and only if f behaves in any other fashion.*

What makes this theorem satisfying is that it perfectly matches up the various series-based descriptions of f about c with the various behavioral (i.e., function-theoretic) descriptions of f near c.

Proof. Consider the Laurent series of f about c,

$$f(z) = g(\zeta) = \sum_{n=-\infty}^{\infty} a_n \zeta^n, \quad \text{where } \zeta = \begin{cases} z - c & \text{if } c \in \mathbb{C}, \\ 1/z & \text{if } c = \infty. \end{cases}$$

The behavior of f near c is the behavior of g near 0.

If the singularity is removable then $a_n = 0$ for all $n < 0$, and so $g(\zeta) \to a_0$ as $\zeta \to 0$, i.e., g is bounded near 0.

If the singularity is a pole of order $N > 0$ then $g(\zeta) = \zeta^{-N} h(\zeta)$ where h is analytic at 0 and $h(0) \neq 0$. This goes to ∞ as $\zeta \to 0$.

If the singularity is essential then by the Casorati-Weierstrass theorem, g is neither bounded nor uniformly large near 0.

Thus the three implications ($\Longrightarrow$) are proved. And since the three behaviors are exclusive and exhaustive, the three implications ($\Longleftarrow$) follow. □