THE RATIO TEST

Consider a complex power series all of whose coefficients are nonzero,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n, \quad a_n \neq 0 \text{ for each } n.$$

Suppose that the limit

$$R = R(f) = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

exists in the extended nonnegative real number system $[0,\infty]$. We show that R is the radius of convergence of f,

> *$f(z)$ converges absolutely on the open disk of radius R about c, and this convergence is uniform on compacta, but $f(z)$ diverges if $|z-c| > R$.*

Not every power series has coefficients that are all nonzero, and even if all the coefficients are nonzero then the limit R needn't exist, so the statement here is only a partial result. For the full story, see this course's related writeup on the radius of convergence formula, involving an idea called the *limit superior.*

We freely take $c = 0$, and we proceed by cases.

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1. The case $0 \le R < \infty$

If $0 < R < \infty$, let z vary through a compact subset K of the open disk of radius R about 0; this open disk is empty for $R = 0$. Thus, for some $r \in (0,1)$,

$$|z| < r^2 R, \quad z \in K.$$

Because $\lim_n |a_n|/|a_{n+1}| = R$, there is a starting index N such that

$$\begin{aligned} rR &\le |a_N|/|a_{N+1}| \\ rR &\le |a_{N+1}|/|a_{N+2}| \\ rR &\le |a_{N+2}|/|a_{N+3}|, \end{aligned}$$

and so on. It follows that

$$\begin{aligned} |a_{N+1}| &\le |a_N|/(rR) \\ |a_{N+2}| &\le |a_{N+1}|/(rR) \le |a_N|/(rR)^2 \\ |a_{N+3}| &\le |a_{N+2}|/(rR) \le |a_N|/(rR)^3, \end{aligned}$$

and in general

$$|a_{N+k}| \leq |a_N|/(rR)^k, \quad k = 0, 1, 2, \dots,$$

from which

$$|a_{N+k}z^{N+k}| \leq |a_N|/(rR)^k \cdot (r^2R)^N (r^2R)^k, \quad k = 0, 1, 2, \dots .$$

Let $C = |a_N|(r^2R)^N$, and now the previous display is

$$|a_{N+k}z^{N+k}| \leq Cr^k, \quad k = 0, 1, 2, \dots .$$

The head of the sum of the absolute values of the terms of the power series satisfies the estimate

$$\sum_{n=0}^{N-1} |a_n z^n| \leq \sum_{n=0}^{N-1} |a_n|(r^2R)^n,$$

and the tail satisfies

$$\sum_{n=N}^{\infty} |a_n z^n| \leq C \sum_{k=0}^{\infty} r^k = \frac{C}{1-r} .$$

So $\sum_{n \geq 0} |a_n z^n|$ converges altogether. The convergence uniform over K because for $M \geq N$,

$$\sum_{n=M}^{\infty} |a_n z^n| \leq Cr^{M-N} \sum_{k=0}^{\infty} r^k = \frac{C}{1-r} r^{M-N},$$

and as M goes to ∞, this goes to 0 independently of where z lies in K.

Now with $0 \leq R < \infty$, suppose that $|z| > R$. Because $\lim_n |a_n|/|a_{n+1}| = R$, there is a starting index N such that

$$|a_{N+k}|/|a_{N+k+1}| < |z|, \quad k = 0, 1, 2, \dots,$$

and so, similarly to above,

$$|a_{N+k}| > |a_N|/|z|^k, \quad k = 1, 2, 3, \dots,$$

from which, with $C = |a_N||z|^N > 0$,

$$|a_{N+k}z^{N+k}| > C, \quad k = 1, 2, 3, \dots .$$

Thus $\sum_{n=0}^{\infty} a_n z^n$ diverges because its terms don't go to 0.

2. The case $R = \infty$

Let z vary through any compact subset K of $\mathbb{C}$. Thus for some $d > 0$,

$$|z| \leq d, \quad z \in K.$$

Because $\lim_n |a_n|/|a_{n+1}| = \infty$, there is a starting index N such that

$$2d \leq |a_{N+k}|/|a_{N+k+1}|, \quad k = 0, 1, 2, \dots,$$

so that

$$|a_{N+k}| \leq |a_N|/(2d)^k, \quad k = 0, 1, 2, \dots,$$

As above, now with $C = |a_N|d^N$,

$$|a_{N+k}z^{N+k}| \leq |a_N|/(2d)^k \cdot d^{N+k} = C/2^k, \quad k = 0, 1, 2, \dots .$$

From here the convergence argument is exactly as before, now with $r = 1/2$. No divergence argument is needed here because the convergence holds everywhere.

3. Absolute convergence implies convergence

Let $\{c_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} |c_n|$ converges. We show that $\sum_{n=0}^{\infty} c_n$ converges.

To say that $\sum_{n=0}^{\infty} |c_n|$ converges is to say that its sequence of partial sums,

$$\{s_N\} = \{\sum_{n=0}^{N} |c_n|\},$$

converges. Thus this sequence of partial sums is Cauchy, meaning that for any $\varepsilon > 0$ there exists a starting index $N_o = N_o(\varepsilon)$ such that $|s_N - s_M| < \varepsilon$ for all $N, M \geq N_o$. That is, for all $N, M \geq N_o$, freely taking $N \geq M$,

$$|c_{M+1}| + |c_{M+2}| + \cdots + |c_N| < \varepsilon.$$

By the triangle inequality it follows that for all $N, M \geq N_o$, again taking $N \geq M$,

$$|c_{M+1} + c_{M+2} + \cdots + c_N| < \varepsilon.$$

This says that the sequence of partial sums,

$$\{t_N\} = \{\sum_{n=0}^{N} c_n\},$$

is Cauchy. Because $\mathbb{C}$ is complete, this sequence therefore converges. That is, $\sum_{n=0}^{\infty} c_n$ converges.

4. Absolutely convergent series can be rearranged

Let $\{c_n\}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} |c_n|$ converges. Let $\{d_n\}$ be a rearrangement of $\{c_n\}$, meaning that

$$\{d_0, d_1, d_2, \dots\} = \{c_{m(0)}, c_{m(1)}, c_{m(2)}, \dots\}$$

where $m : \{0, 1, 2, \dots\} \longrightarrow \{0, 1, 2, \dots\}$ is a bijection. We show that $\sum_{n=0}^{\infty} d_n = \sum_{n=0}^{\infty} c_n$.

For any N there exists a minimal $M = M(N) \geq N$ such that $\{m(0), \dots, m(M)\}$ contains $\{0, \dots, N\}$. Thus, for any $L \geq M$, with the primed summation sign denoting a finite sum in the next display,

$$\left| \sum_{n=0}^{L} d_n - \sum_{n=0}^{L} c_n \right| = \left| {\sum_{n>N}}' (\pm c_n) \right| \leq {\sum_{n>N}}' |c_n|.$$

Because $\sum_{n=0}^{\infty} |c_n|$ converges, ${\sum_{n>N}}' |c_n|$ goes to 0 as N grows. So for large L, $\sum_{n=0}^{L} d_n$ is close to $\sum_{n=0}^{L} c_n$, which is close to $\sum_{n=0}^{\infty} c_n$. That is, $\sum_{n=0}^{\infty} d_n = \sum_{n=0}^{\infty} c_n$.