TERMWISE DERIVATIVES OF COMPLEX FUNCTIONS

This writeup proves a result that has as *one* consequence that any complex power series can be differentiated term-by-term within its disk of convergence. The result has other consequences as well. Recall some results that are already established. One is that the uniform limit of continuous functions is continuous:

Let S be a subset of $\mathbb{C}$. Consider a sequence of continuous functions on S,

$$\{\varphi_0, \varphi_1, \varphi_2, \dots\} : S \longrightarrow \mathbb{C}.$$

Suppose that the sequence converges uniformly on S to a limit function

$$\varphi : S \longrightarrow \mathbb{C}.$$

Then φ is also continuous.

This was shown in the previous writeup on compactness and uniformity. The other result to recall is that we can pass uniform limits through integrals. Specifically, let Ω be a region in $\mathbb{C}$ and let a be any point of Ω. Some closed ball $\overline{B}$ centered at a lies in Ω. Let $\gamma = \partial\overline{B}$ be the boundary circle of $\overline{B}$, traversed once counterclockwise. Suppose that a sequence of continuous functions

$$\{\phi_0, \phi_1, \phi_2, \dots\} : \gamma \longrightarrow \mathbb{C}$$

converges uniformly on γ to a limit $\phi : \gamma \longrightarrow \mathbb{C}$. We know that ϕ must also be continuous. Then as also was shown in a previous writeup, the limit of the integrals is the integral of the limit,

$$\lim_n \int_\gamma \phi_n(\zeta)\, d\zeta = \int_\gamma \phi(\zeta)\, d\zeta.$$

Now we can state and prove our main result. The following theorem is due to Weierstrass.

Let Ω be a region in $\mathbb{C}$. Consider a sequence of differentiable functions on Ω,

$$\{\varphi_0, \varphi_1, \varphi_2, \dots\} : \Omega \longrightarrow \mathbb{C}.$$

Suppose that the sequence converges on Ω to a limit function

$$\varphi : \Omega \longrightarrow \mathbb{C}$$

and that the convergence is uniform on compact subsets of Ω. Then

(1) *The limit function φ is differentiable.*
(2) *The sequence $\{\varphi_n'\}$ of derivatives converges on Ω to the derivative φ' of the limit function.*
(3) *This convergence is also uniform on compact subsets of Ω.*

Proof. First, to show that the limit function φ is continuous, let z be any point of Ω. Some closed ball $\overline{B}$ centered at z lies in Ω, and the convergence of $\{\varphi_n\}$ to φ is uniform on the compact set $\overline{B}$. The restriction of the limit function φ to $\overline{B}$ is therefore continuous, and so φ itself is continuous at the interior point z. Since z is arbitrary, φ is continuous on Ω.

Next, to show that φ is differentiable, let $\gamma = \partial\overline{B}$ be the boundary circle of the closed ball $\overline{B}$ from the previous paragraph, traversed once counterclockwise, and let z be any point inside γ. Consider an auxiliary sequence of functions on γ,

$$\phi_n(\zeta) = \frac{\varphi_n(\zeta)}{\zeta - z}, \quad n = 0, 1, 2, \ldots,$$

with limit

$$\phi(\zeta) = \frac{\varphi(\zeta)}{\zeta - z}.$$

Since $\{\phi_n\}$ converges uniformly on γ to ϕ, we may exchange an integral and a limit,

$$\begin{aligned}\varphi(z) = \lim_n \varphi_n(z) = \lim_n \frac{1}{2\pi i}\int_\gamma \frac{\varphi_n(\zeta)}{\zeta - z}\,d\zeta &= \lim_n \frac{1}{2\pi i}\int_\gamma \phi_n(\zeta)\,d\zeta \\ &= \frac{1}{2\pi i}\int_\gamma \phi(\zeta)\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta.\end{aligned}$$

This shows that the continuous function φ has a Cauchy integral representation, making it differentiable.

Third, use the Cauchy integral representation of derivatives to argue similarly (with a modified auxiliary sequence $\{\phi_n\}$) that the sequence $\{\varphi_n'\}$ of derivatives converges to the derivative φ' of the limit function,

$$\lim_n \varphi_n'(z) = \lim_n \frac{1}{2\pi i}\int_\gamma \frac{\varphi_n(\zeta)}{(\zeta - z)^2}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^2}\,d\zeta = \varphi'(z).$$

Finally, we need to argue that this convergence is uniform on compact subsets of Ω. In the special case that the compact set is the closed ball $\overline{B'}$ having half the radius of the open ball B, let $c > 0$ denote the half-radius, so that

$$|\zeta - z| \geq c \quad \text{for all } \zeta \in \gamma \text{ and } z \in \overline{B'}.$$

It follows by Cauchy's formula for the derivative and the usual estimation techniques that for all $z \in \overline{B'}$,

$$\begin{aligned}|\varphi'(z) - \varphi_n'(z)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{\varphi(\zeta) - \varphi_n(\zeta)}{(\zeta - z)^2}\,d\zeta\right| \\ &\leq \frac{1}{2\pi}\int_\gamma \frac{|\varphi(\zeta) - \varphi_n(\zeta)|}{c^2}\,|d\zeta| \\ &= C \cdot \sup\{|\varphi(\zeta) - \varphi_n(\zeta)| : \zeta \in \gamma\}.\end{aligned}$$

But $\{\varphi_n\}$ converges to φ uniformly on the compact subset γ of Ω. So, given $\varepsilon > 0$, there exists a starting index n_0 such that

$$n \geq n_0 \implies |\varphi'(z) - \varphi_n'(z)| < \varepsilon \quad \text{for all } z \in \overline{B'}.$$

To complete the argument, let K be any compact set of the whole region Ω. About each point a of K there is a ball $B = B_a$ as in the previous discussion. Let B_a' be the ball about a of half the radius of B_a. These balls give an open cover of K,

$$K = \bigcup_{a \in K} \{a\} \subset \bigcup_{a \in K} B_a'.$$

By compactness, K has a finite subcover,

$$K \subset B_{a_1}' \cup \cdots \cup B_{a_k}'.$$

Let $K_j = K \cap \overline{B'_{a_j}}$ for $j = 1, \ldots, k$. Then

$$K = K_1 \cup \cdots \cup K_n,$$

and by the previous paragraph, the convergence of $\{\varphi'_n\}$ to φ' is uniform on each of the finitely-many sets K_j. Consequently the convergence is uniform on K: Given $\varepsilon > 0$, the corresponding global starting index n_0 is the maximum of finitely many local ones. This completes the proof.

As mentioned, the application of the Weierstrass Theorem that we have in mind here is that the functions φ_n are the partial sums of a power series,

$$\varphi_n(z) = \sum_{j=0}^{n} a_j (z-c)^j, \quad n = 0, 1, 2, \ldots$$

while φ is the full power series,

$$\varphi(z) = \sum_{j=0}^{\infty} a_j (z-c)^j.$$

In this case, the result is that any power series can be differentiated term by term within its disk of convergence, and the resulting power series has the same disk of convergence.

For another application of the Weierstrass Theorem, consider the Euler–Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Since $|1/n^s| = 1/n^{\mathrm{Re}(s)}$, the sum converges absolutely on the right half plane $\Omega = \{\mathrm{Re}(s) > 1\}$, and the convergence is uniform on compacta. Thus $\zeta(s)$ is analytic on Ω.

For a third application of the Weierstrass Theorem, let $\Omega = \mathbb{C} - \mathbb{Z}$, a region in $\mathbb{C}$. Define for each $n \in \mathbb{N}$,

$$\varphi_n : \Omega \longrightarrow \mathbb{C}, \qquad \varphi_n(z) = \frac{1}{z} + \sum_{j=1}^{n} \left(\frac{1}{z-j} + \frac{1}{z+j} \right),$$

and define the corresponding limit function

$$\varphi : \Omega \longrightarrow \mathbb{C}, \qquad \varphi(z) = \frac{1}{z} + \sum_{j=1}^{\infty} \left(\frac{1}{z-j} + \frac{1}{z+j} \right).$$

It can be shown that the sequence $\{\varphi_n\}$ converges to φ uniformly on compact subsets of Ω, and so by the Weierstrass Theorem, all derivatives of φ exist on Ω. The series φ can be written as a sum over $\mathbb{Z}$ rather than as a sum of paired terms, but at the cost of introducing modifications to the individual terms to force convergence,

$$\varphi(z) = \frac{1}{z} + \sum_{0 \neq j \in \mathbb{Z}} \left(\frac{1}{z-j} + \frac{1}{j} \right).$$

Its derivative is a similar sum, though no longer with modified terms,

$$\varphi' : \Omega \longrightarrow \mathbb{C}, \qquad \varphi'(z) = -\sum_{j \in \mathbb{Z}} \frac{1}{(z-j)^2}.$$

In fact,

$$\varphi(z) = \pi \cot \pi z \quad \text{and} \quad \varphi'(z) = -\pi^2 \csc \pi z,$$

but showing this takes a little work. Even the periodicity of φ is not obvious, because of the modified terms in its defining sum. For the derivatives of φ, periodicity it is more clear, although even with the modifications gone, the argument relies on the fact that an absolutely convergent sum can be rearranged with no effect on its value.

We will pursue all of this in another writeup.

For a fourth application of the Weierstrass Theorem, let ω_1 and ω_2 be nonzero complex numbers such that ω_1/ω_2 is not real. These two numbers span a lattice in $\mathbb{C}$,

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}.$$

For each $n \in \mathbb{N}$ let

$$\Lambda'_n = \{\omega \in \Lambda : \omega \neq 0,\ |\omega| \leq n\},$$

and let

$$\Lambda' = \{\omega \in \Lambda : \omega \neq 0\}.$$

Let $\Omega = \mathbb{C} - \Lambda$, a region in $\mathbb{C}$. Define for each $n \in \mathbb{N}$,

$$\wp_n : \Omega \longrightarrow \mathbb{C}, \qquad \wp_n(z) = \frac{1}{z^2} + \sum_{\Lambda'_n} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

This is the sequence of partial sums of the *Weierstrass $\wp$-function*,

$$\wp : \Omega \longrightarrow \mathbb{C}, \qquad \wp(z) = \frac{1}{z^2} + \sum_{\Lambda'} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

As with the series for the cotangent, the modifications $-1/\omega^2$ to the summands for $\wp$ are required to make the sum absolutely convergent. Unlike the series for cotangent, this series does not represent a function that is already familiar. It is a genuinely new function arising from complex analysis.

It can be shown that the sequence $\{\wp_n\}$ converges to $\wp$ uniformly on compact subsets of Ω, and so by the Weierstrass Theorem, all derivatives of $\wp$ exist on Ω as similar sums, though no longer with modified terms. In particular we have

$$\wp' : \Omega \longrightarrow \mathbb{C}, \qquad \wp'(z) = -2 \sum_{\Lambda'} \frac{1}{(z-\omega)^3}.$$

The Weierstrass $\wp$-function and its derivatives are *doubly periodic* with respect to the lattice Λ, meaning for example that

$$\wp(z+\omega) = \wp(z) \quad \text{for all } \omega \in \Lambda.$$

For the derivatives of $\wp$ this is fairly clear, using absolute convergence. The double periodicity of $\wp$ itself is not obvious, because of the modified factors in the series, but it follows quickly from the facts that $\wp$ is even and $\wp'$ is doubly periodic.

We may discuss doubly periodic functions later in the course.

Another application of the Weierstrass Theorem in this course will be more abstract, in the proof of a result called the Riemann Mapping Theorem. The proof will construct a sequence of functions that come ever closer to having desired properties, and then the Weierstrass theorem will then guarantee the existence of a limit function, for which the desired properties will hold.

An odd consequence of the academic semester system is that sometimes a first complex analysis course will use the Weierstrass Theorem to prove that power series are analytic and termwise differentiable (even though we have already seen that this result is readily proved without the theorem), but then not get to any other applications of the Weierstrass Theorem, such as the properties of the Weierstrass $\wp$-function. This creates a doubly false impression that the power series result is difficult and that the theorem serves only one purpose.