TERMWISE DERIVATIVES OF POWER SERIES, SANS INTEGRALS

A direct argument, making no reference to integral representation, shows that any complex power series is termwise differentiable in its disk of convergence.

Consider a power series, centered at 0 without loss of generality, and consider also its termwise derivative,

\[ p(z) = \sum_{n=0}^{\infty} a_n z^n, \quad q(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \]

Assume that \( p \) has a positive radius of convergence, and let \( D \) denote its open disk of convergence, the open disk of convergence of \( q \) as well. Let \( z \) be any fixed point of \( D \), so that all points \( \zeta \) close enough to \( z \) also lie in \( D \). We want to show that

\[ \lim_{\zeta \to z} \frac{p(\zeta) - p(z)}{\zeta - z} = q(z). \]

There exists some positive \( r \) less than the radius of convergence of \( p \) such that \(|z| < r\). Suppose also that \(|\zeta| < r\), a condition that holds for all \( \zeta \) close enough to \( z \). For any nonnegative integer \( N \), define the \( N \)-th partial sum and the \( N \)-th error of \( p \),

\[ s_N(p; z) = \sum_{n=0}^{N} a_n z^n, \quad e_N(p; z) = \sum_{n=N+1}^{\infty} a_n z^n, \]

and define \( s_N(q) \) and \( e_N(q) \) similarly. Thus \( p = s_N(p) + e_N(p) \) and \( q = s_N(q) + e_N(q) \). We want a difference to go to 0 as \( \zeta \) goes to \( z \). Decompose it into three pieces,

\[ \frac{p(\zeta) - p(z)}{\zeta - z} - q(z) = A_N(z, \zeta) + B_N(z, \zeta) + C_N(z), \]

where

\[ A_N(z, \zeta) = \frac{s_N(p; \zeta) - s_N(p; z)}{\zeta - z} - s_N(q; z), \]

\[ B_N(z, \zeta) = \frac{e_N(p; \zeta) - e_N(p; z)}{\zeta - z}, \]

\[ C_N(z) = -e_N(q; z). \]

Let \( \epsilon > 0 \) be given. For any fixed \( N \), \( A_N(z, \zeta) \) goes to 0 as \( \zeta \) goes to \( z \), because the polynomial \( s_N(q) \) is the derivative of the polynomial \( s_N(p) \). Also, because \( \zeta^n - z^n = (\zeta - z) \sum_{j=0}^{n-1} \zeta^j z^{n-1-j} \), recalling that \(|z| < r \) and \(|\zeta| < r \),

\[ |B_N(z, \zeta)| = \left| \sum_{n=N+1}^{\infty} a_n \frac{\zeta^n - z^n}{\zeta - z} \right| = \left| \sum_{n=N+1}^{\infty} a_n \sum_{j=0}^{n-1} \zeta^j z^{n-1-j} \right| \leq \sum_{n=N+1}^{\infty} n |a_n| r^{n-1}. \]

The last sum in the previous display is the tail of a convergent series because \( q(r) \) converges absolutely, and so \(|B_N(z, \zeta)| < \epsilon \) if the fixed \( N \) is large enough and \( \zeta \) is close enough to \( z \). Further, for large enough \( N \), we have \(|C_N(z)| < \epsilon \) because
the power series \( q(z) \) converges. Altogether, choosing \( N \) large enough establishes a statement that makes no reference to \( N \):

Given \( \varepsilon > 0 \),

\[
\left| \frac{p(\zeta) - p(z)}{\zeta - z} - q(z) \right| < 3\varepsilon \quad \text{for all } \zeta \text{ close enough to } z.
\]

This is the desired result.

A variant argument is more direct, at the cost of using the difference-of-powers formula twice, as follows. We have

\[
\frac{p(\zeta) - p(z)}{\zeta - z} - q(z) = \sum_{n=1}^{\infty} a_n \left( \frac{\zeta^n - z^n}{\zeta - z} - nz^{n-1} \right).
\]

For \( n = 1 \), the term in parentheses is 0. For \( n \geq 2 \), it is

\[
\frac{\zeta^n - z^n}{\zeta - z} - nz^{n-1} = \left( \sum_{j=0}^{n-1} \zeta^{n-1-j} z^j \right) - nz^{n-1} = \sum_{j=0}^{n-2} (\zeta^{n-1-j} z^j - z^{n-1})
\]

\[
= \sum_{j=0}^{n-2} z^j (\zeta^{n-1-j} - z^{n-1-j}) = (\zeta - z) \sum_{j=0}^{n-2} \zeta^k z^{n-2-k},
\]

and so, again with \( |z| < r \) and \( |\zeta| < r \) where \( r \) is less than the radius of convergence,

\[
\left| \frac{p(\zeta) - p(z)}{\zeta - z} - q(z) \right| \leq |\zeta - z| \sum_{n=1}^{\infty} |a_n| n^2 r^{n-2}.
\]

The series on the right side of the inequality converges, and so the left side goes to 0 as \( \zeta \) goes to \( z \). That is, \( p'(z) \) exists and equals \( q(z) \). This is the desired result.