TERMWISE DERIVATIVES OF POWER SERIES, SANS INTEGRALS

A direct argument, making no reference to integral reprsentation, shows that any complex power series is termwise differentiable in its disk of convergence. (This writeup is taken nearly verbatim from a writeup by Paul Garrett.)

Consider a power series, centered at 0 without loss of generality, and consider also its termwise derivative,

$$p(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad q(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Assume that p has a positive radius of convergence, and let D denote its open disk of convergence, the open disk of convergence of q as well. Let z and ζ be any distinct points of D. Then we have

$$\frac{p(\zeta) - p(z)}{\zeta - z} - q(z) = \sum_{n=1}^{\infty} a_n \left(\frac{z^n - \zeta^n}{z - \zeta} - n z^{n-1} \right).$$

For $n = 1$, the term in parentheses is 0. For $n \geq 2$, it is

$$\begin{aligned}
\frac{z^n - \zeta^n}{z - \zeta} - n z^{n-1} &= \left(\sum_{j=0}^{n-1} z^{n-1-j} \zeta^j \right) - n z^{n-1} = \left(\sum_{j=1}^{n-1} z^{n-1-j} \zeta^j \right) - (n-1) z^{n-1} \\
&= \sum_{j=1}^{n-1} (z^{n-1-j} \zeta^j - z^{n-1}) = \sum_{j=1}^{n-1} z^{n-1-j} (\zeta^j - z^j) \\
&= \sum_{j=1}^{n-1} z^{n-1-j} (\zeta - z) \sum_{k=0}^{j-1} \zeta^k z^{j-1-k} = (\zeta - z) \sum_{j=1}^{n-1} \sum_{k=0}^{j-1} \zeta^k z^{n-2-k} \\
&= (\zeta - z) \sum_{k=0}^{n-2} \sum_{j=k+1}^{n-1} \zeta^k z^{n-2-k} = (\zeta - z) \sum_{k=0}^{n-2} (n-k-1) \zeta^k z^{n-2-k}.
\end{aligned}$$

Let $\rho = \max\{|z|, |\zeta|\} < r$, where r is the radius of convergence of p. We have shown that

$$\left| \frac{z^n - \zeta^n}{z - \zeta} - n z^{n-1} \right| < |z - \zeta| n^2 \rho^{n-2},$$

and therefore that

$$\left| \frac{p(\zeta) - p(z)}{\zeta - z} - q(z) \right| < |z - \zeta| \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2}.$$

The series on the right side of the inequality converges, and so the left side goes to 0 as ζ goes to z. That is, $p'(z)$ exists and equals $q(z)$. This is the desired result.