TERMWISE DERIVATIVES OF POWER SERIES, SANS INTEGRALS

A direct argument, making no reference to integral representation, shows that any complex power series is termwise differentiable in its disk of convergence. (This writeup is taken nearly verbatim from a writeup by Paul Garrett.)

Consider a power series, centered at 0 without loss of generality, and consider also its termwise derivative,

\[ p(z) = \sum_{n=0}^{\infty} a_n z^n, \quad q(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \]

Assume that \( p \) has a positive radius of convergence, and let \( D \) denote its open disk of convergence, the open disk of convergence of \( q \) as well. Let \( z \) and \( \zeta \) be any distinct points of \( D \). Then we have

\[
\frac{p(\zeta) - p(z)}{\zeta - z} - q(z) = \sum_{n=1}^{\infty} a_n \left( \frac{\zeta^n - z^n}{\zeta - z} - n z^{n-1} \right).
\]

For \( n = 1 \), the term in parentheses is 0. For \( n \geq 2 \), it is

\[
\frac{\zeta^n - z^n}{\zeta - z} - n z^{n-1} = \left( \sum_{j=0}^{n-1} \zeta^{n-1-j}z^j \right) - n z^{n-1} = \sum_{j=0}^{n-2} (\zeta^{n-1-j}z^j - z^{n-1})
\]

\[
= \sum_{j=0}^{n-2} z^j(\zeta^{n-1-j} - z^{n-1-j}) = \sum_{j=0}^{n-2} (\zeta - z)^j \sum_{k=0}^{n-2-j} \zeta^k z^{n-2-j-k}
\]

\[
= (\zeta - z) \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} \zeta^k z^{n-2-k} = (\zeta - z) \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-k} \zeta^k z^{n-2-k}
\]

\[
= (\zeta - z) \sum_{k=0}^{n-2} (n-1-k)\zeta^k z^{n-2-k}.
\]

Let \( \rho = \max\{|z|,|\zeta|\} < r \), where \( r \) is the radius of convergence of \( p \). We have shown that

\[
\left| \frac{\zeta^n - z^n}{\zeta - z} - n z^{n-1} \right| < |\zeta - z| n^2 \rho^{n-2},
\]

and therefore that

\[
\left| \frac{p(\zeta) - p(z)}{\zeta - z} - q(z) \right| < |\zeta - z| \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2}.
\]

The series on the right side of the inequality converges, and so the left side goes to 0 as \( \zeta \) goes to \( z \). That is, \( p'(z) \) exists and equals \( q(z) \). This is the desired result.