

TERMWISE DERIVATIVES OF POWER SERIES

We have passed a z -derivative through a ζ -integral to establish Cauchy's integral representation formula for derivatives, and we have passed an n -limit of sums through a ζ -integral to establish the power series representation of complex-differentiable functions. To establish the differentiability of complex power series, we need to pass a z -derivative through an n -limit of sums. This limit-exchange reduces to the two limit-exchanges that we have already carried out.

Let

$$p(z) = \sum_{j=0}^{\infty} a_j(z-c)^j$$

be a power series, and let its partial sums be

$$p_n(z) = \sum_{j=0}^n a_j(z-c)^j, \quad n = 0, 1, 2, \dots$$

Assume that p has a positive radius of convergence, and let D denote its disk of convergence. Let z be any point of D . Some closed disk \overline{B} about z of positive radius lies in D . Since \overline{B} is compact and p converges uniformly on \overline{B} , the restriction of p to \overline{B} is continuous, and so p itself is continuous on the interior of \overline{B} ; in particular p is continuous at z . Since z is arbitrary, p is continuous on all of D .

Again let z be any point of D . Let γ be a small circle around z in D . Then p is continuous and hence integrable on γ .

The calculation proceeds as follows. By the definition of the infinite sum as the limit of the finite sums, because each finite sum is differentiable and therefore has integral representation, and because the integrand converges uniformly on γ , so that the limit passes through the integral,

$$p(z) = \lim_{n \rightarrow \infty} p_n(z) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{p_n(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\gamma} \frac{p(\zeta) d\zeta}{\zeta - z}.$$

The power series is now part of the integrand, but it no longer involves z . The convergence

$$p(\zeta) \cdot \frac{\frac{1}{\zeta - \bar{z}} - \frac{1}{\zeta - z}}{\bar{z} - z} \xrightarrow{\bar{z} \rightarrow z} p(\zeta) \cdot \frac{1}{(\zeta - z)^2}$$

occurs uniformly on γ , allowing us to take the z -derivative by passing it through the integral,

$$p'(z) = \frac{d}{dz} \left[\frac{1}{2\pi i} \int_{\gamma} \frac{p(\zeta) d\zeta}{\zeta - z} \right] = \frac{1}{2\pi i} \int_{\gamma} \frac{p(\zeta) d\zeta}{(\zeta - z)^2}.$$

Pass the limit back out through the integral, and then use the integral representation of the derivative to get

$$p'(z) = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{p_n(\zeta) d\zeta}{(\zeta - z)^2} = \lim_{n \rightarrow \infty} p'_n(z).$$

This last limit exists because the one before it exists.

Summarizing, the power series can be differentiated termwise within its disk of convergence,

$$\frac{d}{dz} \sum_{j=0}^{\infty} a_j (z - c)^j = \sum_{j=1}^{\infty} j a_j (z - c)^{j-1}.$$