CONSEQUENCES OF POWER SERIES REPRESENTATION

1. The Uniqueness Theorem

Theorem 1.1 (Uniqueness)**.** *Let $\Omega \subset \mathbb{C}$ be a region, and consider two analytic functions*

$$f, g : \Omega \longrightarrow \mathbb{C}.$$

Suppose that S is a subset of Ω that has a limit-point $p \in \Omega$. (The limit-point p need not lie in S.) Suppose that $f = g$ on S. Then $f = g$.

For example, the unique analytic function on $\mathbb{C}$ that vanishes at 1, 1/2, 1/4, 1/8, 1/16, ... is the zero function. For another example, the extensions by power series of e^x, $\sin x$, $\cos x$, and $\log x$ from their domains in $\mathbb{R}$ to analytic functions on $\mathbb{C}$ (on $\mathbb{C}$ minus the negative real axis for log) are the unique possible such extensions.

Proof. We may assume that $g = 0$. That is, we may assume that $f = 0$ on S. And we may assume that $p = 0$.

Let B be the largest ball about 0 in Ω. Possibly $r = +\infty$, but in any case $r > 0$. The power series representation of f at 0 is

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \quad z \in B.$$

Because 0 is a limit-point of S, some sequence $\{z_n\}$ in S satisfies the conditions

$$\lim_{n\to\infty} \{z_n\} = 0, \qquad z_n \neq 0 \text{ for all } n,$$

Thus, since f is continuous at p and since $f = 0$ on S,

$$a_0 = f(0) = \lim_{n\to\infty} \{f(z_n)\} = \lim_{n\to\infty} \{0\} = 0.$$

So $f(z) = zg(z)$ on B, where $g(z) = a_1 + a_2 z + z_3 z^2 + \cdots$. Note that $g = 0$ on S. Similarly to a moment ago,

$$a_1 = g(0) = \lim_{n\to\infty} \{g(z_n)\} = \lim_{n\to\infty} \{0\} = 0.$$

Repeating the argument shows that every coefficient of the power series expansion of f about 0 vanishes. That is, the power series expansion is 0. Thus $f = 0$ on B.

But we want f to be identically zero on all of Ω. So let q be any point of Ω. Since Ω is connected and open in $\mathbb{C}$, a little topology shows that it is path-connected, and the connecting paths can be taken to be rectifiable. The general topological principle here is that *connected and locally path-connected implies path-connected*, and in our context the connecting paths can be taken to be rectifiable by metric properties of $\mathbb{C}$.

Thus some rectifiable path γ in the region Ω connects p to q. As argued in the writeup about Cauchy's Theorem, since γ is compact some ribbon about it lies in the region as well (here we need only an open ribbon),

$$R = \bigcup_{z\in\gamma} B(z, \rho) \subset \Omega.$$

Form a chain of finitely many disks of radius ρ, with their centers spaced at most distance ρ apart along γ, starting at p and ending at q. (This is where it matters that γ is rectifiable.) Each consecutive pair of disks overlaps on a set S having the center of the second disk as a limit-point. Since f is identically zero on the first disk, the argument just given shows that is identically zero on the second disk as well, and so on up to last disk, so that in particular $f(q) = 0$. This completes the proof. $\square$

A consequence of the Uniqueness Theorem is

Corollary 1.2. *An analytic function $f : \Omega \longrightarrow \mathbb{C}$ that is not identically zero has isolated zeros in any compact subset K of Ω, and hence only finitely many zeros in any such K. More generally, if f is not constant then on any compact subset K of Ω and for any value $a \in \mathbb{C}$, f has only finitely many a-points, meaning points where f takes the value a.*

This holds because any infinite subset S of a compact subset K has a limit-point in K by the Bolzano–Weierstrass Theorem. So if $f = a$ everywhere on S then $f = a$ identically on Ω.

2. The Maximum Principle

Theorem 2.1. *Suppose that $f : \Omega \longrightarrow \mathbb{C}$ is analytic. Then either $|f|$ assumes no maximum on Ω or f is constant.*

Proof. Suppose that $|f|$ assumes a maximum at some point $c \in \Omega$. That is,

$$|f(z)| \leq |f(c)| \quad \text{for all } z \in \Omega.$$

Some disk $B = B(c, r)$ where $r > 0$ lies in Ω. For any ρ satisfying $0 < \rho < r$, let γ_ρ be the circle about c of radius ρ, and compute that

$$|f(c)| = \left| \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(z)\, dz}{z - c} \right| \leq \frac{1}{2\pi} \int_{\gamma_\rho} \frac{|f(z)|\, |dz|}{\rho} \leq \frac{1}{2\pi\rho} \sup_{z \in \gamma_\rho} \{|f(z)|\} 2\pi\rho \leq |f(c)|.$$

Since the chain of inequalities starts and ends at the same value, all of the inequalities must be equalities, so that $|f| = |f(c)|$ on γ_ρ. Since $\rho \in (0, r)$ is arbitrary, in fact $|f| = |f(c)|$ on B. By a homework problem, f is constant on B, and so by the Uniqueness Theorem, f is constant on Ω. $\square$

A consequence of the Maximum Principle is

Corollary 2.2. *If $f : \Omega \longrightarrow \mathbb{C}$ is analytic and K is a compact subset of Ω then $\max_{z \in K}\{|f(z)|\}$ is assumed on the boundary of K.*

3. Liouville's Theorem

Theorem 3.1 (Liouville's Theorem)**.** *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic and bounded. Then f is constant.*

Proof. The power series representation of f at 0 is valid for all of $\mathbb{C}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}.$$

Let M bound $|f|$. Cauchy's Inequality says that for any $r > 0$, and any $n \in \mathbb{N}$,

$$|a_n| < \frac{M}{r^n}.$$

Since r can be arbitrarily large this proves that $a_n = 0$ for $n \geq 1$, i.e., $f(z) = a_0$ for all z. □

4. The Fundamental Theorem of Algebra

Theorem 4.1 (Fundamental Theorem of Algebra)**.** *Let $p(z)$ be a nonconstant polynomial with complex coefficients. Then p has a complex root.*

Proof. We may take

$$p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j, \quad n \geq 1.$$

Note that for all z such that $|z| \geq 1$,

$$\left| \sum_{j=0}^{n-1} a_j z^j \right| \leq C|z|^{n-1} \quad \text{where } C = \sum_{j=0}^{n-1} |a_j|.$$

It follows that for all z such that $|z| > C + 1$,

$$|p(z)| \geq |z|^n - C|z|^{n-1} > |z|^{n-1} \geq 1.$$

Now suppose that $p(z)$ has no complex root. Then the function $f(z) = 1/p(z)$ is entire. The function $f(z)$ is bounded on the compact set $\overline{B(0, C+1)}$, and it satisfies

$$|f(z)| < 1 \quad \text{for all } z \text{ such that } |z| > C + 1.$$

Therefore $f(z)$ is entire and bounded, making it constant by Liouville's Theorem, and this make the original polynomial $p(z)$ constant as well. The proof is complete by contraposition. □

As a corollary, any nonconstant polynomial of degree $n \geq 1$ factors down to linear terms,

$$p(z) = c \prod_{j=1}^{n} (z - r_j), \quad r_1, \ldots, r_n \in \mathbb{C}.$$

There may be repetitions among the roots.

5. Polynomial Behavior at Infinity

Theorem 5.1. *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function. For each positive integer n, f is a polynomial of degree n if and only if*

$$\lim_{|z| \to +\infty} \frac{|f(z)|}{|z|^n} \text{ exists and is a nonzero constant.}$$

Proof. If $f(z)$ is a polynomial of degree n,

$$f(z) = a_n z^n + \sum_{j=0}^{n-1} a_j z^j, \quad a_n \neq 0,$$

then basic estimates show that for $C = \sum_{j=0}^{n-1} |a_j|$ and $|z| \geq 1$,

$$|a_n|\,|z|^n - C|z|^{n-1} \leq |f(z)| \leq |a_n|\,|z|^n + C|z|^{n-1},$$

so that

$$\lim_{|z|\to+\infty} \frac{|f(z)|}{|z|^n} = |a_n|.$$

And in fact the estimates further show that

$$\lim_{|z|\to+\infty} \frac{|f(z)|}{|z|^m} = \begin{cases} +\infty & \text{if } m < n, \\ |a_n| & \text{if } m = n, \\ 0 & \text{if } m > n. \end{cases}$$

Conversely, if

$$\lim_{|z|\to+\infty} \frac{|f(z)|}{|z|^n} = c \quad \text{where } c \neq 0$$

then

$$|f(z)| \leq 2c|z|^n \quad \text{for all } z \text{ such that } |z| \text{ is large enough.}$$

Using Cauchy's Inequality as on a homework problem, it follows that f is a polynomial of degree at most n. And its degree can't be less than n by the three-case formula above. □

6. The Casorati–Weierstrass Theorem

An entire function that is not a polynomial is called *transcendental.* (This is a special-case usage: the term "transcendental" has a more general meaning in a broader context.)

By the homework problem mentioned above, if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is entire and transcendental, then for any positive integer n there is a sequence $\{z_j\}$ in $\mathbb{C}$ with

$$\lim_j \{|z_j|\} = +\infty \quad \text{and} \quad \lim_j \left\{ \frac{|f(z_j)|}{|z_j|^n} \right\} = +\infty.$$

That is, f grows faster than any polynomial on some sequence of z-values. But the behavior of f as $|z|$ gets large is more extreme than this, as follows.

Theorem 6.1 (Casorati–Weierstrass Theorem). *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be entire and transcendental. Given any three values $\varepsilon > 0$, $r > 0$, and $c \in \mathbb{C}$, the set*

$$\{z \in \mathbb{C} : |z| > r\}$$

contains points z such that $|f(z) - c| < \varepsilon$.

The import of the theorem is that an entire transcendental function behaves wildly as its inputs tend to infinity. Its outputs don't simply tend to infinity very quickly, they also go essentially everywhere.

Proof. If f has infinitely many c-points p_1, p_2, p_3, ... then we are done: the condition $|p_n| \leq r$ can hold for only finitely many of them, as in the corollary of the Uniqueness Theorem, for otherwise f would be identically c rather than transcendental

If f has only finitely many c-points p_1, ..., p_n then the power series expansion of $f - c$ is

$$f(z) - c = \prod_{j=1}^{n} (z - p_j)^{e_j} \cdot g(z),$$

where $e_1, \ldots, e_n$ are positive integers and g is an entire function that doesn't vanish. Also, g is not constant since f is transcendental. Its reciprocal,

$$h = 1/g : \mathbb{C} \longrightarrow \mathbb{C},$$

is entire and nonconstant, and it has no zeros, making it transcendental. Let $e = \sum_{j=1}^{n} e_j$. The hypothetical condition

$$|h(z)| \leq |z|^{e+1} \text{ for all } z \text{ outside some disk}$$

would force h to be a polynomial. Since h is not a polynomial, it follows by contraposition that the negation of the hypothetical condition must hold instead,

$$\text{for any } r > 0 \text{ there is some } z \text{ such that } |z| > r \text{ and } |h(z)| > |z|^{e+1}.$$

Thus there is a sequence $\{z_n\}$ in $\mathbb{C}$ such that

$$\lim_n \{|z_n|\} = +\infty \quad \text{and} \quad \lim_n \left\{ \left| \frac{h(z_n)}{z_n^e} \right| \right\} = +\infty. \tag{1}$$

But for z away from $p_1, \ldots, p_n$ we have

$$\frac{h(z)}{z^e} = \frac{\prod_{j=1}^{n}(z - p_j)^{e_j}}{z^e(f(z) - c)}.$$

The numerator has degree e, so that in general,

$$\lim_{|z|\to\infty} \frac{\prod_{j=1}^{n}(z - p_j)^{e_j}}{z^e} = 1.$$

Hence it follows from (1) that

$$\lim_n \{|z_n|\} = +\infty \quad \text{and} \quad \lim_n \left\{ \frac{1}{|f(z_n) - c|} \right\} = +\infty.$$

That is,

$$\lim_n \{|z_n|\} = +\infty \quad \text{and} \quad \lim_n \{f(z_n)\} = c,$$

and this is the desired result. □

In this context, the following result deserves mention.

Theorem 6.2 (Picard's Theorem)**.** *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be entire and transcendental. Given any value $r > 0$, the set*

$$\{f(z) : |z| > r\}$$

is all of $\mathbb{C}$ except at most one point.

This is a very strong theorem, and its proof is beyond us for now. Until we prove it, do not solve problems by citing Picard's Theorem.

As an example of Picard's Theorem, consider the geometry of the complex exponential function. However large its input z is required to be in absolute value, a horizontal strip of such inputs of height 2π exists, and its outputs are all of the punctured plane.