SUM OF LOGARITHMS, LOGARITHM OF PRODUCT

We begin with an easy observation. Let $z_1, z_2, \ldots \in \mathbb{C}$. Suppose that the sequence of their partial sums converges,

$$\left\{ \sum_{n=1}^{N} z_n \right\} = \{s_N\} \xrightarrow{N} s.$$

Because the exponential is a continuous homomorphism from $(\mathbb{C}, +)$ to $(\mathbb{C}^\times, \cdot)$, the sequence of partial products of the exponentials converges correspondingly,

$$\left\{ \prod_{n=1}^{N} e^{z_n} \right\} = \{p_N\} \xrightarrow{N} p \quad \text{where } p = e^s.$$

Briefly, if $\sum_n z_n = s$ then $\prod_n e^{z_n} = e^s$.

We want a converse result. Let $z_1, z_2, \ldots \in \mathbb{C}^\times$. Suppose that the sequence of their partial products converges in $\mathbb{C}^\times$ (convergence to 0 is not convergence in the multiplicative group $\mathbb{C}^\times$),

$$\left\{ \prod_{n=1}^{N} z_n \right\} = \{p_N\} \xrightarrow{N} p \neq 0.$$

We will show that for any appropriately sensible chosen branch of the complex logarithm, the sequence of partial sums of the logarithms converges to some logarithm of the product,

$$\left\{ \sum_{n=1}^{N} \log z_n \right\} = \{s_N\} \xrightarrow{N} s, \quad \text{where } s \in \log p + 2\pi i \mathbb{Z}.$$

But because no branch of the logarithm is a homomorphism, no claim can be made that $s = \log p$ for the branch of log that is being used. Briefly, if $\prod_n z_n = p \neq 0$ then $\sum \log z_n \in \log p + 2\pi i \mathbb{Z}$.

To begin the argument, recall that we have a sequence $z_1, z_2, \ldots$ in $\mathbb{C}^\times$ such that the sequence $\{\prod_{n=1}^{N} z_n\} = \{p_N\}$ of partial products converges to some $p$ in $\mathbb{C}^\times$. Note that the multiplicative analogue of the $n$th term test applies to the sequence of partial products: Because $\{p_N\}$ converges to $p$ in $\mathbb{C}^\times$, the individual terms $z_N$ must converge to 1,

$$\left\{ \frac{p_N}{p_{N-1}} \right\} \xrightarrow{N} \frac{p}{p} = 1.$$

Also, the disk $D$ about $p$ of radius $|p|/2$ misses 0, and all but finitely many of the partial products $p_N$ lie in $D$, and the disk $D'$ about 1 of radius 1/2 misses 0, and all but finitely many of the $z_N$ lie in $D'$. Take a branch cut that misses $D$ and misses $D'$ and misses the finitely many terms $z_N$ and partial products $p_N$ outside of the two disks. Specifically, the branch cut can be chosen as a ray from the origin, not $\mathbb{R}_{\geq 0}$, and then the logarithm can be chosen so that $\log 1 = 0$. 

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Because the sequence \( \{ p_N \} \) converges to \( p \) in the domain of our branch of the logarithm, which is continuous, also
\[
\{ \log p_N \} \xrightarrow{N} \log p.
\]
But our concern is the convergence of the sum \( \sum_n \log z_n \), whose partial sums \( \sum_{n=1}^{N} \log z_n \) need not equal \( \log p_N \), and so we need to consider how they differ. Because \( \{ \log p_N \} \) converges, we have
\[
\{ \log p_N - \log p_{N-1} \} \xrightarrow{N} 0.
\]
Also, note that
\[
\begin{align*}
\log p_N - \log p_{N-1} &= \log(p_{N-1}z_N) - \log p_{N-1} \\
&= \log p_{N-1} + \log z_N + 2\pi ik_N - \log p_{N-1} \quad \text{for some } k_N \in \mathbb{Z} \\
&= \log z_N + 2\pi ik_N.
\end{align*}
\]
So now we have
\[
\{ \log z_N + 2\pi ik_N \} \xrightarrow{N} 0.
\]
But \( \{ z_N \} \xrightarrow{N} 1 \) and \( \log 1 = 0 \), so in fact
\[
\{ k_N \} \xrightarrow{N} 0.
\]
Because each \( k_N \) is an integer, there exists a starting index \( N_o \) such that
\[
k_N = 0 \quad \text{for all } N \geq N_o.
\]
This shows that the logarithm does behave homomorphically on the partial products of large index,
\[
\log p_N = \log p_{N-1} + \log z_N \quad \text{for all } N \geq N_o.
\]
And one integer \( k \) captures the non-homomorphic behavior up to the starting index where the homomorphic behavior begins,
\[
\sum_{n=1}^{N_o} \log z_n = \log p_{N_o} + 2\pi ik \quad \text{for some } k \in \mathbb{Z}.
\]
Thus the sequence of partial sums of the logarithms of the sequence terms is, from the \( N_o \)th term onwards,
\[
\{ \sum_{n=1}^{N} \log z_n \}_{N \geq N_o} = \{ \log p_N \}_{N \geq N_o} + 2\pi ik.
\]
We know that the sequence \( \{ \log p_N \} \) converges to \( \log p \), and so the claim is established,
\[
\{ \sum_{n=1}^{N} \log z_n \} \xrightarrow{N} \log p + 2\pi ik.
\]