# SKETCH OF PICARD'S THEOREM

The writeup sketches a proof of one version of Picard's Theorem: *Any entire function that misses at least two points,*

$$f : \mathbf{C} \longrightarrow \mathbf{C} - \{p, q\},$$

*is constant.* After an affine shift, we may assume that $p = 0$ and $q = 1728$.

Let $\mathcal{H}$ denote the complex upper half-plane, and let $\Gamma$ denote the group $\mathrm{SL}_2(\mathbf{Z})$ of 2-by-2 matrices with integral entries and determinant 1. Then $\Gamma$ acts on $\mathcal{H}$ via fractional linear transformations,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \tau \in \mathcal{H}.$$

Let $\zeta_3 = e^{2\pi i/3}$. There is a complex-analytic covering map

$$j : \mathcal{H} - (\Gamma(i) \cup \Gamma(\zeta_3)) \longrightarrow \mathbf{C} - \{0, 1728\}.$$

Specifically,

$$j = 1728 \frac{g_2^3}{g_2^3 - 27g_3^2},$$

where for any $\tau \in \mathcal{H}$,

$$g_2(\tau) = 60 \cdot \sum_{(c,d)}{}' \frac{1}{(c\tau + d)^4}, \qquad g_3(\tau) = 140 \cdot \sum_{(c,d)}{}' \frac{1}{(c\tau + d)^6}.$$

Here the sums are over pairs $(c, d) \in \mathbf{Z}^2$ and the primes mean to exclude $(0, 0)$ as a summand.

Given an entire function $f : \mathbf{C} \longrightarrow \mathbf{C} - \{0, 1728\}$, fix a point $z \in \mathbf{C}$. Consider any path from 0 to $z$,

$$\beta_z : [0, 1] \longrightarrow \mathbf{C}, \quad \beta(0) = 0, \ \beta(1) = z.$$

Compose with $f$ to get a path in $\mathbf{C} - \{0, 1728\}$ with the corresponding endpoints,

$$\gamma_z = f \circ \beta_z : [0, 1] \longrightarrow \mathbf{C} - \{0, 1728\}, \quad \gamma_z(0) = f(0), \ \gamma_z(1) = f(z).$$

Let $w_0 = f(0)$, and choose any $\tau_0 \in \mathcal{H} - (\Gamma(i) \cup \Gamma(\zeta_3))$ such that $j(\tau_0) = w_0$. By general topology, the path $\gamma_z$ has a unique lift $\delta_z$ starting at $\tau_0$,

$$\delta_z : [0, 1] \longrightarrow \mathcal{H} - (\Gamma(i) \cup \Gamma(\zeta_3)), \quad \delta_z(0) = \tau_0, \ j \circ \delta_z = \gamma_z.$$

Then *the endpoint $\delta_z(1)$ of $\delta_z$ depends only on the endpoint $z$ of the initial path $\beta_z$ from 0 to $z$, not on the choice of $\beta_z$ itself.* Indeed, since any two paths from 0 to $z$ are homotopic in $\mathbf{C}$, the homotopy type of $\beta_z$ depends only on $z$. Homotopy passes through continuous functions, so the homotopy type of the path $\gamma_z = f \circ \beta_z$ from $w_0 = f(0)$ to $f(z)$ in $\mathbf{C} \longrightarrow \mathbf{C} - \{0, 1728\}$ depends only on $z$ as well. From topology, the endpoint of the lift $\delta_z$ in $\mathcal{H} - (\Gamma(i) \cup \Gamma(\zeta_3))$ starting at $\tau_0$ now depends only on $z$ in turn, as desired. (In general, paths $\gamma$ from $f(0) = w_0$ to $f(z)$ in $\mathbf{C} - \{0, 1728\}$ need not be homotopic, and so their lifts $\delta$ starting at $\tau_0$ can have different endpoints. That is, the argument of this paragraph relies on the function $f$

being continuous and having a simply connected domain. But we haven't yet used the full force of the hypothesis that $f$ is entire.)

With the endpoint of the path $\delta_z$ known to depend only on $z$, define now the function taking each $z$ to the corresponding endpoint,

$$F : \mathbf{C} \longrightarrow \mathcal{H}, \qquad F(z) = \delta_z(1).$$

Since the covering map $j$ is analytic, so are its local inverses, and so the lifted endpoint $F(z) = \delta_z(1)$ is analytic as a function of the nonlifted endpoint $f(z) = \gamma_z(1)$, which itself is analytic as a function of $z$. That is, $F$ is entire.

Being an entire function into the upper half plane, $F$ is constant. But the only way for the lifting process that produces $F$ to give a constant is for $f$ to be constant. This completes the (sketched) argument.