A FAR-REACHING LITTLE INTEGRAL

Let

- r be any positive real number, and γ_r be the circle of radius r centered at the origin, traversed once counterclockwise,
- n be any integer, and $f_n(z) = z^n$. This function is undefined at $z = 0$ if n is negative.

The natural parameterization of γ_r is

$$\gamma_r : [0, 2\pi] \longrightarrow \mathbb{C}, \qquad \gamma_r(t) = re^{it} = z,$$

and so the integral of f_n over γ_r is

$$\begin{aligned}
\int_{\gamma_r} f_n(z)\,\mathrm{d}z &= \int_{t=0}^{2\pi} (re^{it})^n \, d(re^{it}) \\
&= \int_{t=0}^{2\pi} r^n e^{int}\, ire^{it}\,\mathrm{d}t \\
&= ir^{n+1} \int_{t=0}^{2\pi} e^{i(n+1)t}\,\mathrm{d}t \\
&= ir^{n+1} \cdot \begin{cases} 2\pi & \text{if } n = -1, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

That is, the integral

$$\int_{\gamma_r} z^n\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise} \end{cases}$$

is independent of r and nearly independent of n.

The preceding formula has enormous consequences. For example, naively assuming that some function f has a representation in integer powers of z,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

and naively assuming that the sum passes through integration over γ_r, it follows that integrating f over $\gamma = \gamma_r$ (for any suitable $r > 0$) picks off the coefficient a_{-1} of $1/z$ in f and ignores everything else,

$$\frac{1}{2\pi i} \int_\gamma f(z)\,\mathrm{d}z = a_{-1}.$$

Making these ideas precise requires some care, and there are some subtleties, but things pretty much work out as the calculation here suggests.