A FAR-REACHING LITTLE INTEGRAL

Let
- \( r \) be any positive real number, and \( \gamma_r \) be the circle of radius \( r \) centered at the origin, traversed once counterclockwise,
- \( n \) be any integer, and \( f_n(z) = z^n \). This function is undefined at \( z = 0 \) if \( n \) is negative.

The natural parameterization of \( \gamma_r \) is
\[ \gamma_r : [0, 2\pi] \to \mathbb{C}, \quad \gamma_r(t) = re^{it} = z, \]
and so the integral of \( f_n \) over \( \gamma_r \) is
\[
\int_{\gamma_r} f_n(z) \, dz = \int_{t=0}^{2\pi} (re^{it})^n \, d(re^{it}) \\
= \int_{t=0}^{2\pi} r^n e^{int} ire^{it} \, dt \\
= i r^{n+1} \int_{t=0}^{2\pi} e^{i(n+1)t} \, dt \\
= i r^{n+1} \cdot \begin{cases} 
2\pi & \text{if } n = -1, \\
0 & \text{otherwise} 
\end{cases} \\
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2\pi i & \text{if } n = -1, \\
0 & \text{otherwise}. 
\end{cases}
\]

That is, the integral
\[
\int_{\gamma_r} z^n \, dz = \begin{cases} 
2\pi i & \text{if } n = -1, \\
0 & \text{otherwise} 
\end{cases}
\]
is independent of \( r \) and nearly independent of \( n \).

The preceding formula has enormous consequences. For example, naively assuming that some function \( f \) has a representation in integer powers of \( z \),
\[ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \]
and naively assuming that the sum passes through integration over \( \gamma_r \), it follows that integrating \( f \) over \( \gamma = \gamma_r \) (for any suitable \( r > 0 \)) picks off the coefficient \( a_{-1} \) of \( 1/z \) in \( f \) and ignores everything else,
\[
\frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = a_{-1}.
\]
Making these ideas precise requires some care, and there are some subtleties, but things pretty much work out as the calculation here suggests.