MORERA'S THEOREM

Let Ω be a region. Recall some ideas:

- Cauchy's theorem says that if γ is a simple closed rectifiable curve in Ω, and if f is an analytic function on an open superset of γ and its interior, then
$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$
- In consequence of Cauchy's theorem, Cauchy's integral representation formula says that if γ is a simple closed rectifiable curve in Ω, and if f is an analytic function on an open superset of γ and its interior, then for every point z in the interior of γ,
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}\,.$$
- Differentiation under the integral sign shows that if a continuous function $f : \Omega \longrightarrow \mathbb{C}$ has the integral representation of the previous bullet, then f is $\mathcal{C}^\infty$ on Ω, and its derivatives also have integral representation; specifically, for any γ and z as in the previous bullet,
$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{k+1}}\,, \quad k = 0, 1, 2, \dots\,.$$
In particular, an analytic function on Ω is $\mathcal{C}^\infty$ on Ω. From here, a geometric series argument shows that in fact an analytic function on Ω is $\mathcal{C}^\omega$ on Ω.

1. Statement

Morera's theorem is a partial converse of Cauchy's theorem, as follows.

Theorem 1.1 (Morera)**.** *Let Ω be a region, and let $f : \Omega \longrightarrow \mathbb{C}$ be continuous. Suppose that*
$$\int_\gamma f(z)\,\mathrm{d}z = 0 \quad \textit{for all simple closed rectifiable curves } \gamma \textit{ in } \Omega.$$
Then f is analytic on Ω.

Proof. We need to show that f' exists on Ω. Fix any point z_o in Ω. The following function is well defined:
$$F : \Omega \longrightarrow \mathbb{C}, \qquad F(z) = \int_{z_o}^{z} f(\zeta)\,\mathrm{d}\zeta,$$
where the integral is taken along any rectifiable curve from z_o to z. For any $z \in \Omega$ and all small enough nonzero $h \in \mathbb{C}$ we have, integrating over the line segment from

z to $z+h$,

$$\begin{aligned}\frac{F(z+h)-F(z)}{h} &= \frac{\int_z^{z+h} f(\zeta)\,\mathrm{d}\zeta}{h} \\ &= \frac{\int_z^{z+h} (f(z)+o(1))\,\mathrm{d}\zeta}{h} \\ &= f(z) + \frac{1}{h}\int_z^{z+h} o(1)\,\mathrm{d}\zeta.\end{aligned}$$

Given any $\varepsilon > 0$, the integrand satisfies $|o(1)| \le \varepsilon$ if h is small enough, and so the absolute value of the integral is at most $\varepsilon|h|$ for all such h. That is, for every $\varepsilon > 0$, the difference quotient is within ε of $f(z)$ for all small enough nonzero h. This means precisely that $F'(z)$ exists and equals $f(z)$. Because z is any point of Ω this shows that

$$F' = f \quad \text{on } \Omega.$$

Because F is analytic on Ω, it is $\mathcal{C}^\infty$ on Ω, and in particular its second derivative exists on Ω. That is, f' exists on Ω. □

The proof of Morera's theorem shows that if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all simple closed rectifiable curves γ in Ω then $f = F'$ for some analytic $F : \Omega \longrightarrow \mathbb{C}$. The converse of this statement is true as well, by the complex fundamental theorem of calculus, and so we have a partial converse to Morera's theorem, that if an analytic function f is a derivative then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all simple closed rectifiable curves γ in Ω. However, the full converse of Morera's theorem is not true, the function $f(z) = 1/z$ on $\Omega = \mathbb{C} - \{0\}$ being the standard counterexample. Although f is analytic on Ω, it is not a derivative there, and its integral over the unit circle is nonzero.

2. Consequence: the converse of Cauchy's theorem

By contrast, an essential converse of Cauchy's theorem is true, thanks to Morera's theorem. Again let Ω be a region, and let $f : \Omega \longrightarrow \mathbb{C}$ be continuous. Cauchy's theorem says that if $f : \Omega \longrightarrow \mathbb{C}$ is analytic then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all simple closed rectifiable curves γ in Ω such that the interior of γ lies in Ω.

Now rather than assume that f is analytic, assume instead that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all simple closed rectifiable curves γ in Ω such that the interior of γ lies in Ω. For each point z of Ω, let B_z denote the largest open disk about z in Ω, and note that every simple closed rectifiable curve γ in B_z is such that the interior of γ lies in B_z, and so $\int_\gamma f(z)\,\mathrm{d}z = 0$. Thus f on B_z satisfies the hypothesis of Morera's theorem, and so Morera's theorem says that f' exists on B_z. In particular $f'(z)$ exists. Because z is an arbitrary point of Ω, this shows that f is analytic on Ω.

3. Summary

Let Ω be a region, and let γ always denote a simple closed rectifiable curve in Ω. Consider a continuous function $f : \Omega \longrightarrow \mathbb{C}$. Then

$$\begin{array}{ccc} \begin{pmatrix} f = F' \\ \text{for some } F : \Omega \longrightarrow \mathbb{C} \end{pmatrix} & \Longleftrightarrow & \begin{pmatrix} \int_\gamma f(z)\,\mathrm{d}z = 0 \\ \text{for all } \gamma \end{pmatrix} \\ \Big\Downarrow & & \Big\Downarrow \\ \big(f' \text{ exists on } \Omega\big) & \Longleftrightarrow & \begin{pmatrix} \int_\gamma f(z)\,\mathrm{d}z = 0 \\ \text{for all } \gamma \text{ with } \mathrm{int}(\gamma) \subset \Omega \end{pmatrix} \end{array}$$

and

- the top $\Longrightarrow$ is by the complex fundamental theorem of integral calculus
- the top $\Longleftarrow$ is Morera's theorem
- the left downward implication follows from integral representation and differentiation under the integral sign, i.e., $f' = F''$ exists because F' exists
- the right downward implication is clear
- the bottom $\Longrightarrow$ is Cauchy's theorem
- the bottom $\Longleftarrow$ follows from Morera's theorem by a small argument.

And again, the example to keep in mind is $f(z) = 1/z$ on $\Omega = \mathbb{C} - \{0\}$, a differentiable function that is not a derivative and whose integral around the unit circle is $2\pi i$.

If a region Ω is *simply connected,* meaning that $\mathrm{int}(\gamma) \subset \Omega$ for all simple closed rectifiable curves γ in Ω, then the two conditions on the right side of the diagram above are the same, and so:

On a simply connnected domain Ω, f' exists if and only if $f = F'$ for some F.

What is particular to simply connected domains is that having a derivative implies having an antiderivative.