

THE RADIUS OF CONVERGENCE FORMULA

A real sequence $\{x_n\}$ may have no limit, but it always has so-called *upper* and *lower limits*, denoted

$$\limsup x_n \quad \text{and} \quad \liminf x_n.$$

Each of these can assume the extended real values $+\infty$ and $-\infty$. The notation $\limsup x_n$ literally means

$$\lim_{n \rightarrow \infty} \left\{ \sup_{m \geq n} x_m \right\},$$

(note that this is the limit of a monotonically decreasing sequence), but this definition is cumbersome and shouldn't be parsed literally while one is in the middle of computing. The idea is that the \limsup is the largest limit-point of the sequence $\{x_n\}$. When the \limsup is finite, it is characterized as follows:

- (1) Given $\varepsilon > 0$, $x_n < \limsup x_n + \varepsilon$ for all n greater than some N .
- (2) Given $\varepsilon > 0$, $x_n > \limsup x_n - \varepsilon$ for infinitely many n .

(Note the asymmetry between the two conditions, a consequence of the asymmetry built into the notion of \limsup .) Suitable adjustments need to be made for the infinite cases, but they are straightforward. If the \limsup is $-\infty$, then condition (1) becomes:

- (1') Given $r \in \mathbf{R}$, $x_n < r$ for all n greater than some N ,

while condition (2) becomes irrelevant. If the \limsup is $+\infty$, then condition (1) becomes irrelevant and condition (2) becomes:

- (2') Given $r \in \mathbf{R}$, $x_n > r$ for infinitely many n .

And $\liminf x_n$ is handled similarly. A good exercise is to show that for real sequences $\{x_n\}$ and $\{y_n\}$,

$$\begin{aligned} \liminf x_n + \liminf y_n &\leq \liminf(x_n + y_n) \\ &\leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n \end{aligned}$$

excepting the undefined case $\infty - \infty$. Working through this should show when the various " \leq " signs are equality or strict inequality.

The reason that we care about these notions is that given a complex power series,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - c)^n,$$

there is an explicit formula for its radius of convergence:

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

The result is that $f(z)$ converges absolutely on the open disk of radius R about c , and this convergence is uniform on compacta, but $f(z)$ diverges if $|z - c| > R$.

To show this, suppose first that R is finite. Let K be a nonempty compact subset of the open disk of radius R about c . (Thus we are also assuming that $R > 0$, since otherwise there is no such K , and so we are assuming that $\limsup \sqrt[n]{|a_n|}$ is finite.)

Then a maximum value of $|z - c|$ exists as z varies through K , and this maximum value is strictly less than R . That is, for some ε satisfying $0 < \varepsilon \leq R/2$,

$$|z - c| \leq R - 2\varepsilon \quad \text{for all } z \in K.$$

Note that the expression $1/(R - \varepsilon)$ is greater than $1/R$, and so it takes the form

$$\frac{1}{R - \varepsilon} = \limsup \sqrt[n]{|a_n|} + \varepsilon', \quad \text{where } \varepsilon' > 0.$$

Therefore the radius of convergence formula and the characterizing property (1) of lim sup combine to show that

$$\sqrt[n]{|a_n|} < \frac{1}{R - \varepsilon} \quad \text{for all } n \text{ greater than some } N.$$

It follows that

$$|a_n(z - c)^n| < \left(\frac{R - 2\varepsilon}{R - \varepsilon} \right)^n \quad \text{for all } n \text{ greater than } N.$$

This shows that the power series $f(z) = \sum_n a_n(z - c)^n$ converges absolutely, by comparison with the geometric series $\sum_n ((R - 2\varepsilon)/(R - \varepsilon))^n$. Since the geometric sum converges at a rate that depends only on ε , the convergence is uniform over the compact subset K of z -values with which we are working.

If instead $R = +\infty$, then let K be any compact subset of \mathbf{C} . There is some positive number d such that

$$|z - c| < d \quad \text{for all } z \in K.$$

Now $\limsup \{ \sqrt[n]{|a_n|} \} = 0$, and so by the characterizing property (1) of lim sup,

$$\sqrt[n]{|a_n|} < \frac{1}{2d} \quad \text{for all } n \text{ greater than some } N.$$

Therefore

$$|a_n(z - c)^n| < \frac{1}{2^n} \quad \text{for all } n \text{ greater than } N.$$

Again the power series $f(z) = \sum_n a_n(z - c)^n$ converges absolutely, by comparison with the geometric series $\sum_n 1/2^n$. And again, the convergence is uniform over the compact subset K of z -values with which we are working.

On the other hand, suppose that R is finite and $|z - c| > R$. Then $|z - c| = R + \varepsilon$ for some $\varepsilon > 0$. If R is positive then $\limsup \sqrt[n]{|a_n|}$ is finite. In this case, the expression $1/(R + \varepsilon)$ is less than the lim sup, and so it takes the form

$$\frac{1}{R + \varepsilon} = \limsup \sqrt[n]{|a_n|} - \varepsilon', \quad \text{where } \varepsilon' > 0.$$

Therefore the radius of convergence formula and the characterizing property (2) of lim sup combine to show that

$$\sqrt[n]{|a_n|} > \frac{1}{R + \varepsilon} \quad \text{for infinitely many } n.$$

If $R = 0$ then the lim sup is $+\infty$, and so this same condition holds by the alternate characterizing property (2') for this case. Regardless, it follows from the previous display that

$$|a_n(z - c)^n| > \left(\frac{R + \varepsilon}{R + \varepsilon} \right)^n = 1 \quad \text{for infinitely many } n.$$

This shows that $f(z)$ diverges, by the n th term test.

In examples, either the ratio test or the formula

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right| \quad \text{if the limit exists}$$

will often be easier to use than the lim sup formula for the radius of convergence. But the point is that for the ratio test or the displayed formula to give the answer, a certain limit must exist in the first place, whereas the lim sup formula always works, making it handy for general arguments,