THE RADIUS OF CONVERGENCE FORMULA

Every complex power series,

\[ f(z) = \sum_{n=0}^{\infty} a_n(z - c)^n, \]

has a radius of convergence, a nonnegative-real or infinite, \( R = R(f) \in [0, +\infty] \), that describes the convergence of the series, as follows.

\( f(z) \) converges absolutely on the open disk of radius \( R \) about \( c \), and this convergence is uniform on compacta, but \( f(z) \) diverges if \( |z - c| > R \).

The radius of convergence has an explicit formula (notation to be explained below):

\[ R = \frac{1}{\limsup_n \sqrt[n]{|a_n|}} \]

For example, this formula shows immediately that every complex power series \( f(z) = \sum_{n\geq 0} a_n(z - c)^n \) and its formal derivative \( g(z) = \sum_{n\geq 1} na_n(z - c)^{n-1} \) have the same radius of convergence, because \( \lim_n \sqrt[n]{n} = 1 \).

1. LIMIT SUPERIOR AND LIMIT INFERIOR OF A REAL SEQUENCE

Let a real sequence \( \{x_n\} \) be given.

The sequence \( \{x_n\} \) may have no limit, but it always has a limit superior and a limit inferior (also called its upper and lower limits), denoted

\[ \limsup x_n \text{ and } \liminf x_n. \]

Each of these can assume the extended real values \(+\infty\) and \(-\infty\). The notation \( \limsup x_n \) literally means

\[ \lim_{n \to \infty} \left\{ \sup_{m \geq n} x_m \right\}, \]

(note that this is the limit of a monotonically decreasing sequence), but this definition is cumbersome and shouldn’t be parsed literally while one is in the middle of computing. The idea is that the upper limit is the largest limit-point of the sequence \( \{x_n\} \), and similarly for the lower limit.

The condition that a real sequence \( \{x_n\} \) essentially precedes a real number \( r \) is written and defined as follows,

\[ \{x_n\} \preceq r \quad \text{if} \quad x_n < r \quad \text{for all but finitely many } n. \]

The complementary condition, that \( \{x_n\} \) not essentially precede \( r \), is

\[ \{x_n\} \npreceq r \quad \text{if} \quad r \leq x_n \quad \text{for infinitely many } n. \]

Note that if \( \{x_n\} \) doesn’t essentially precede \( r \), it doesn’t follow that \( \{x_n\} \) essentially exceeds \( r \). Some basic observations to be made are as follows.
• If \( \{x_n\} \preceq r \) and \( r < s \) then \( \{x_n\} \preceq s \).
• If \( \{x_n\} \npreceq r \) and \( q < r \) then \( \{x_n\} \npreceq q \).
• If \( \{x_n\} \npreceq q \) and \( \{x_n\} \preceq r \) then \( q < r \).

Thus, for a given real sequence \( \{x_n\} \), introducing the sets
\[
A = \{ r \in \mathbb{R} : \{x_n\} \npreceq r \}, \quad B = \{ r \in \mathbb{R} : \{x_n\} \preceq r \},
\]
we have \( \mathbb{R} = A \sqcup B \) with \( a < b \) for all \( a \in A \) and \( b \in B \), and so exactly one of three possibilities holds in consequence:

• \( A = \mathbb{R} \) and \( B = \emptyset \), i.e., \( \{x_n\} \npreceq r \) for all \( r \in \mathbb{R} \). This means that for each real number \( r \), we have \( r \leq x_n \) for infinitely many \( n \). In some sense, a subsequence of \( \{x_n\} \) converges to \( +\infty \). This condition is written
  \[
  \limsup x_n = +\infty.
  \]

• \( A = \emptyset \) and \( B = \mathbb{R} \), i.e., \( \{x_n\} \preceq r \) for all \( r \in \mathbb{R} \). This means that for each real number \( r \), we have \( x_n < r \) for all but finitely many \( n \). In some sense, \( \{x_n\} \) converges to \( -\infty \), and so no subsequence of \( \{x_n\} \) converges to any real value or to \( +\infty \). This condition is written
  \[
  \limsup x_n = -\infty.
  \]

• \( A \) and \( B \) are nonempty. In this case, the least upper bound \( L \) of \( A \) is also the greatest lower bound of \( B \), and a subsequence of \( \{x_n\} \) converges to \( L \), and no subsequence of \( \{x_n\} \) converges to any real value greater than \( L \) or to \( +\infty \). This condition is written
  \[
  \limsup x_n = L.
  \]

The bound \( L \) is the limit superior or upper limit of \( \{x_n\} \).

We reiterate. When \( \limsup x_n \) is finite, its characterizing properties are as follows:

1. If \( \limsup x_n < r \) then \( x_n < r \) for all but finitely many \( n \).
2. If \( r < \limsup x_n \) then \( r \leq x_n \) for infinitely many \( n \).

Suitable adjustments need to be made for the infinite cases, as follows. If \( \limsup x_n = -\infty \), then condition (1) becomes:

(1') If \( r \in \mathbb{R} \) then \( x_n < r \) for all but finitely many \( n \),

while condition (2) becomes irrelevant. If the \( \limsup \) is \( +\infty \), then condition (1) becomes irrelevant and condition (2) becomes:

(2') If \( r \in \mathbb{R} \) then \( r \leq x_n \) for infinitely many \( n \).

The limit inferior is handled similarly. The exercise to familiarize oneself with these ideas is to show that for real sequences \( \{x_n\} \) and \( \{y_n\} \),

\[
\liminf x_n + \liminf y_n \leq \liminf (x_n + y_n) \\
\leq \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n
\]

excepting the undefined case \( +\infty - \infty \). Also, one should get a feel for when the various \( \preceq \) signs are equality or strict inequality.
2. Radius of Convergence

Reiterating the main result to be shown in this writeup, any given complex power series,

\[ f(z) = \sum_{n=0}^{\infty} a_n (z - c)^n, \]

has a radius of convergence,

\[ R = \frac{1}{\limsup \sqrt[n]{|a_n|}}. \]

Again, the result is that \( f(z) \) converges absolutely on the open disk of radius \( R \) about \( c \), and this convergence is uniform on compacta, but \( f(z) \) diverges if \( |z - c| > R \). We now establish this.

We may take \( c = 0 \). Suppose first that \( R \) is finite. Let \( K \) be a nonempty compact subset of the open disk of radius \( R \). (Thus we are also assuming that \( R > 0 \), because otherwise there is no such \( K \), and so we are assuming that \( \limsup \sqrt[n]{|a_n|} \) is finite.) Then a maximum value of \( |z| \) exists as \( z \) varies through \( K \), and this maximum value is strictly less than \( R \). That is, for some \( \varepsilon \) satisfying \( 0 < \varepsilon \leq R/2 \),

\[ |z| \leq R - 2\varepsilon \quad \text{for all } z \in K. \]

Note that the quantity \( 1/(R - \varepsilon) \) is greater than \( 1/R = \limsup \sqrt[n]{|a_n|} \), and so the characterizing property (1) of \( \limsup \) shows that

\[ \sqrt[n]{|a_n|} < \frac{1}{R - \varepsilon} \quad \text{for all } n \text{ greater than some } N. \]

It follows that

\[ |a_n z^n| < \left( \frac{R - 2\varepsilon}{R - \varepsilon} \right)^n \quad \text{for all } n \text{ greater than } N. \]

This shows that the power series \( f(z) = \sum_n a_n z^n \) converges absolutely, by comparison with the geometric series \( \sum_n \left( (R - 2\varepsilon)/(R - \varepsilon) \right)^n \). Because the geometric sum converges at a rate that depends only on \( \varepsilon \), the convergence is uniform over the compact subset \( K \) of \( z \)-values with which we are working.

If instead \( R = +\infty \), then let \( K \) be any compact subset of \( \mathbb{C} \). There is some positive number \( d \) such that

\[ |z| < d \quad \text{for all } z \in K. \]

Now \( \limsup \sqrt[n]{|a_n|} = 0 \), and so by the characterizing property (1) of \( \limsup \),

\[ \sqrt[n]{|a_n|} < \frac{1}{2d} \quad \text{for all } n \text{ greater than some } N. \]

Therefore

\[ |a_n z^n| < \frac{1}{2^n} \quad \text{for all } n \text{ greater than } N. \]

Again the power series \( f(z) = \sum_n a_n z^n \) converges absolutely, by comparison with the geometric series \( \sum_n 1/2^n \). And again, the convergence is uniform over the compact subset \( K \) of \( z \)-values with which we are working.

On the other hand, suppose that \( R \) is finite and \( |z| > R \). Then \( |z| = R + \varepsilon \) for some \( \varepsilon > 0 \). If \( R \) is positive then \( \limsup \sqrt[n]{|a_n|} \) is finite. In this case, the expression
1/(R + \varepsilon) is less than the lim sup, and so the radius of convergence formula and the characterizing property (2) of lim sup combine to show that
\[
\frac{1}{R + \varepsilon} \leq \sqrt[n]{|a_n|} \quad \text{for infinitely many } n.
\]
If R = 0 then the lim sup is +\infty, and so this same condition holds by the alternate characterizing property (2') for this case. Regardless, it follows from the previous display that
\[
1 = \left( \frac{R + \varepsilon}{R + \varepsilon} \right)^n \leq |a_n z^n| \quad \text{for infinitely many } n.
\]
This shows that f(z) diverges, by the nth term test.

3. Comments

In examples, either the ratio test or the formula
\[
R = \lim \left| \frac{a_n}{a_{n+1}} \right| \quad \text{if the limit exists}
\]
will often be easier to use than the lim sup formula for the radius of convergence. But the point is that for the ratio test or the displayed formula to give the answer, a certain limit must exist in the first place, whereas the lim sup formula always works, making it handy for general arguments,

The radius expressions 1/\lim sup \sqrt[n]{|a_n|} and \lim |a_n/a_{n+1}| in this handout are the reciprocals of the usual expressions from the root test and the ratio test of calculus.
\[
\lim sup \sqrt[n]{|a_n|} \quad \text{and} \quad \lim \left| \frac{a_{n+1}}{a_n} \right|
\]
The issue is that the absolute-terms of our power series are not |a_n| but rather |a_n z^n| (again taking c = 0). Thus the relevant root test and ratio test conditions are
\[
\lim sup \sqrt[n]{|a_n|} |z| < 1 \quad \text{and} \quad \lim \left| \frac{a_{n+1}}{a_n} \right| |z| < 1,
\]
and solving for |z| indeed turns the usual formulas upside down,
\[
|z| < \frac{1}{\lim sup \sqrt[n]{|a_n|}} \quad \text{and} \quad |z| < \lim \left| \frac{a_n}{a_{n+1}} \right|.
\]