THE RADIUS OF CONVERGENCE FORMULA

Every complex power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n,$$

has a *radius of convergence*, nonnegative-real or infinite,

$$R = R(f) \in [0, +\infty],$$

that describes the convergence of the series, as follows.

> *$f(z)$ converges absolutely on the open disk of radius R about c, and this convergence is uniform on compacta, but $f(z)$ diverges if $|z-c| > R$.*

The radius of convergence has an explicit formula (notation to be explained below):

$$\boxed{R = \frac{1}{\limsup \sqrt[n]{|a_n|}}}$$

1. Limit Superior and Limit Inferior of a Real Sequence

Let a real sequence $\{x_n\}$ be given.

The sequence $\{x_n\}$ may have no limit, but it always has a *limit superior* and a *limit inferior* (also called its *upper* and *lower limits*), denoted

$$\limsup x_n \quad \text{and} \quad \liminf x_n.$$

Each of these can assume an extended real value $+\infty$ or $-\infty$. The notation $\limsup x_n$ literally means

$$\lim_{n\to\infty} \left\{ \sup_{m \geq n} x_m \right\}$$

(note that this is the limit of a monotonically decreasing sequence), but this definition is cumbersome and shouldn't be parsed literally while one is in the middle of computing. The idea is that the upper limit is the largest limit of a subsequence of $\{x_n\}$, and similarly for the lower limit.

The condition that a real sequence $\{x_n\}$ *essentially precedes* a real number r is written and defined as follows,

$$\{x_n\} \prec r \qquad \text{if} \quad x_n < r \text{ for all but finitely many } n.$$

The complementary condition, that $\{x_n\}$ does not essentially precede r, is

$$\{x_n\} \not\prec r \qquad \text{if} \quad r \leq x_n \text{ for infinitely many } n.$$

Note that if $\{x_n\}$ doesn't essentially precede r, it doesn't follow that $\{x_n\}$ essentially exceeds r; here the definition of *essentially exceeds* is left unwritten but it should be understood. Some basic observations to be made are as follows.

- If $\{x_n\} \prec r$ and $r \leq s$ then $\{x_n\} \prec s$.
- If $\{x_n\} \not\prec r$ and $q \leq r$ then $\{x_n\} \not\prec q$.

- If $\{x_n\} \not\prec q$ and $\{x_n\} \prec r$ then $q < r$.

Thus, for a given real sequence $\{x_n\}$, introducing the sets

$$A = \{r \in \mathbb{R} : \{x_n\} \not\prec r\}, \qquad B = \{r \in \mathbb{R} : \{x_n\} \prec r\},$$

we have $\mathbb{R} = A \sqcup B$ with $a < b$ for all $a \in A$ and $b \in B$, and so exactly one of three possibilities holds in consequence:

- $A = \mathbb{R}$ and $B = \emptyset$, i.e., $\{x_n\} \not\prec r$ for all $r \in \mathbb{R}$. This means that for each real number r, we have $r \leq x_n$ for infinitely many n. In some sense, a subsequence of $\{x_n\}$ converges to $+\infty$. This condition is written
$$\limsup x_n = +\infty.$$
- $A = \emptyset$ and $B = \mathbb{R}$, i.e., $\{x_n\} \prec r$ for all $r \in \mathbb{R}$. This means that for each real number r, we have $x_n < r$ for all but finitely many n. In some sense, $\{x_n\}$ converges to $-\infty$ and no subsequence of $\{x_n\}$ converges to any real value or to $+\infty$. This condition is written
$$\limsup x_n = -\infty.$$
- A and B are nonempty. In this case, the least upper bound L of A is also the greatest lower bound of B, and a subsequence of $\{x_n\}$ converges to L, and no subsequence of $\{x_n\}$ converges to any real value greater than L or to $+\infty$. This condition is written
$$\limsup x_n = L.$$
The bound L is the *limit superior* or *upper limit* of $\{x_n\}$.

So when $\limsup x_n$ is finite, its characterizing properties are as follows:

(1) If $\limsup x_n < r$ then $x_n < r$ for all but finitely many n.
(2) If $r < \limsup x_n$ then $r \leq x_n$ for infinitely many n.

Suitable adjustments need to be made for the infinite cases. If $\limsup x_n = -\infty$, then condition (1) becomes:

(1′) If $r \in \mathbb{R}$ then $x_n < r$ for all but finitely many n,

while condition (2) becomes irrelevant. If $\limsup x_n = +\infty$, then condition (1) becomes irrelevant and condition (2) becomes:

(2′) If $r \in \mathbb{R}$ then $r \leq x_n$ for infinitely many n.

The limit inferior can be handled similarly, or it can be defined as

$$\liminf\, x_n = -\limsup\,(-x_n).$$

An exercise to familiarize oneself with these ideas is to show that for real sequences $\{x_n\}$ and $\{y_n\}$,

$$\begin{aligned}\liminf\, x_n + \liminf\, y_n &\leq \liminf(x_n + y_n)\\ &\leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,\end{aligned}$$

excepting the undefined case $+\infty - \infty$. Also, one should get a feel for when the various "$\leq$" signs are equality or strict inequality. For example, for the last inequality, there is nothing to prove unless $\limsup\, x_n$ and $\limsup\, y_n$ are both finite and $\limsup(x_n + y_n) \neq -\infty$. Let B_x denote the set B from above for the sequence $\{x_n\}$, and similarly for B_y and B_{x+y}. Thus B_x and B_y are nonempty and bounded below with greatest lower bounds $\limsup\, x_n$ and $\limsup\, y_n$, and the set B_{x+y} is not all of $\mathbb{R}$, so if it is nonempty then it is bounded below with greatest lower bound

$\limsup(x_n + y_n)$. Consider any $r \in B_x$ and $s \in B_y$. Because $x_n < r$ for all but finitely many n, and $y_n < s$ for all but finitely many n, also $x_n + y_n < r + s$ for all but finitely many n. That is, $r + s \in B_{x+y}$. In particular B_{x+y} is nonempty, and now $\limsup(x_n + y_n) = \inf B_{x+y} \leq r + s$. Take infima over r and s to get $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$, the desired inequality.

Another exercise is that if $\{x_n\}$ and $\{y_n\}$ are nonnegative real sequences, and $\lim x_n$ exists and is positive, then $\limsup(x_n y_n) = \lim x_n \cdot \limsup y_n$.

2. Radius of Convergence

Reiterating the main result to be shown in this writeup, any given complex power series,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - c)^n,$$

has a radius of convergence,

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

Again, the result is that $f(z)$ converges absolutely on the open disk of radius R about c, and this convergence is uniform on compacta, but $f(z)$ diverges if $|z-c| > R$. We now establish this.

We may take $c = 0$. Suppose first that R is finite. Let K be a nonempty compact subset of the open disk of radius R. (Thus we are also assuming that $R > 0$, because otherwise there is no such K, and so we are assuming that $\limsup \sqrt[n]{|a_n|}$ is finite.) Then a maximum value of $|z|$ exists as z varies through K, and this maximum value is strictly less than R. That is, for some ε satisfying $0 < \varepsilon \leq R/2$,

$$|z| \leq R - 2\varepsilon \quad \text{for all } z \in K.$$

Note that the quantity $1/(R - \varepsilon)$ is greater than $1/R = \limsup \sqrt[n]{|a_n|}$, and so the characterizing property (1) of $\limsup$ shows that

$$\sqrt[n]{|a_n|} < \frac{1}{R - \varepsilon} \quad \text{for all } n \text{ greater than some } N.$$

It follows that

$$|a_n z^n| < \left(\frac{R - 2\varepsilon}{R - \varepsilon}\right)^n \quad \text{for all } n \text{ greater than } N.$$

This shows that the power series $f(z) = \sum_n a_n z^n$ converges absolutely, by comparison with the geometric series $\sum_n ((R - 2\varepsilon)/(R - \varepsilon))^n$. Because the geometric sum converges at a rate that depends only on ε, the convergence is uniform over the compact subset K of z-values with which we are working.

If instead $R = +\infty$, then let K be any compact subset of $\mathbb{C}$. There is some positive number d such that

$$|z| < d \quad \text{for all } z \in K.$$

Now $\limsup \sqrt[n]{|a_n|} = 0$, and so by the characterizing property (1) of $\limsup$,

$$\sqrt[n]{|a_n|} < \frac{1}{2d} \quad \text{for all } n \text{ greater than some } N.$$

Therefore

$$|a_n z^n| < \frac{1}{2^n} \quad \text{for all } n \text{ greater than } N.$$

Again the power series $f(z) = \sum_n a_n z^n$ converges absolutely, by comparison with the geometric series $\sum_n 1/2^n$. And again, the convergence is uniform over the compact subset K of z-values with which we are working.

On the other hand, suppose that R is finite and $|z| > R$. Then $|z| = R + \varepsilon$ for some $\varepsilon > 0$. If R is positive then $\limsup \sqrt[n]{|a_n|}$ is finite. In this case, the expression $1/(R+\varepsilon)$ is less than the lim sup, and so the definition of R and the characterizing property (2) of lim sup combine to show that

$$\frac{1}{R+\varepsilon} \leq \sqrt[n]{|a_n|} \quad \text{for infinitely many } n.$$

If $R = 0$ then the lim sup is $+\infty$, and so this same condition holds by the alternate characterizing property (2′) for this case. Regardless, it follows from the previous display that

$$1 = \left(\frac{R+\varepsilon}{R+\varepsilon}\right)^n \leq |a_n z^n| \quad \text{for infinitely many } n.$$

This shows that $f(z)$ diverges, by the nth term test.

3. Comments

In examples, either the ratio test or the formula

$$R = \lim \left|\frac{a_n}{a_{n+1}}\right| \quad \text{if the limit exists}$$

will often be easier to use than the lim sup formula for the radius of convergence. But the point is that for the ratio test or the displayed formula to give the answer, a certain limit must exist in the first place, whereas the lim sup formula always works, making it handy for general arguments.

The radius expressions $1/\limsup \sqrt[n]{|a_n|}$ and $\lim |a_n/a_{n+1}|$ in this handout are the reciprocals of the usual expressions from the root test and the ratio test of calculus,

$$\limsup \sqrt[n]{|a_n|} \qquad \text{and} \qquad \lim \left|\frac{a_{n+1}}{a_n}\right|.$$

The reason that the formulas turn upside-down is that the absolute-terms of our power series are not $|a_n|$ but rather $|a_n z^n|$ (again taking $c = 0$). Thus the relevant root test and ratio test conditions are

$$\limsup \sqrt[n]{|a_n|}\,|z| < 1 \qquad \text{and} \qquad \lim \left|\frac{a_{n+1}}{a_n}\right| |z| < 1,$$

and solving for $|z|$ indeed inverts the usual formulas,

$$|z| < \frac{1}{\limsup \sqrt[n]{|a_n|}} \qquad \text{and} \qquad |z| < \lim \left|\frac{a_n}{a_{n+1}}\right|.$$

The second exercise given at the end of section 1 shows that the termwise derivative of a power series has the same radius of convergence as the power series. Indeed, we may multiply the termwise derivative by z with no effect on the radius of convergence, and the resulting coefficients are na_n, and so $\limsup \sqrt[n]{|na_n|} = \lim \sqrt[n]{n} \cdot \limsup \sqrt[n]{|a_n|} = \limsup \sqrt[n]{|a_n|}$; the well known fact that $\lim \sqrt[n]{n} = 1$ is quickly shown by setting $\sqrt[n]{n} = 1 + \epsilon_n$, so that $n = (1 + \epsilon_n)^n > \binom{n}{2}\epsilon_n^2$ and thus $\epsilon_n^2 \le n/\binom{n}{2} \to 0$.

4. Continuity

Abel's elementary proof that complex power series are termwise differentiable in their disk of convergence incidentally shows that they are continuous there as well. However, less elementary proofs, e.g., using Cauchy's integral representation formula, or the Fundamental Theorem of Calculus, tacitly use the continuity. The continuity is a consequence of the uniform convergence of a power series on compact subsets of its disk of convergence. Indeed, let a sequence $\{\varphi_n\} : X \longrightarrow \mathbb{C}$ of complex-valued functions on any subset X of $\mathbb{C}$ converge uniformly to a limit function $\varphi : X \longrightarrow \mathbb{C}$. Thus, given $\varepsilon > 0$, there exists some N such that $|\varphi(x) - \varphi_N(x)| < \varepsilon$ for all $x \in X$. For any given $z \in X$, there exists $\delta_N(z) > 0$ such that also $|\varphi_N(z') - \varphi_N(z)| < \varepsilon$ for all $z' \in X$ such that $|z' - z| < \delta_N$. Thus, for all $z' \in X$ such that $|z' - z| < \delta_N(z)$,

$$|\varphi(z') - \varphi(z)| \le |\varphi(z') - \varphi_N(z')| + |\varphi_N(z') - \varphi_N(z)| + |\varphi_N(z) - \varphi(z)| < 3\epsilon.$$

Especially, φ can be a complex power series and $\{\varphi_n\}$ the sequence of its truncations. For any point z in its disk D of convergence, and any small closed disk X about z in D, the convergence of $\{\varphi_n\}$ to φ on X is uniform, and so the argument applies to show that φ is continuous at z.