PASSING LIMITS THROUGH INTEGRALS

1. A General Lemma

Let Ω be a region in $\mathbb{C}$, and let $\gamma : I \longrightarrow \Omega$ be a rectifiable curve. By a small abuse of notation, the symbol γ will also denote the trace of the curve. Let

$$\{\varphi_n\} : \gamma \longrightarrow \mathbb{C}$$

be a sequence of integrable functions converging uniformly to an integrable function

$$\varphi : \gamma \longrightarrow \mathbb{C}.$$

For example, if each φ_n is continuous then it is integrable, and the uniform convergence then guarantees that φ is continuous and hence integrable as well. Then

$$\lim_{n\to\infty} \int_\gamma \varphi_n(\zeta)\,\mathrm{d}\zeta = \int_\gamma \varphi(\zeta)\,\mathrm{d}\zeta.$$

To prove this, let $\varepsilon > 0$ be given. We may assume that γ has positive length. There exists a starting index n_0 such that

$$n \geq n_0 \implies |\varphi(\zeta) - \varphi_n(\zeta)| < \frac{\varepsilon}{\text{length}(\gamma)} \qquad \text{for all } \zeta \in \gamma.$$

It follows that for all $n \geq n_0$,

$$\begin{aligned}
\left| \int_\gamma \varphi(\zeta)\,\mathrm{d}\zeta - \int_\gamma \varphi_n(\zeta)\,\mathrm{d}\zeta \right| &= \left| \int_\gamma (\varphi(\zeta) - \varphi_n(\zeta))\,\mathrm{d}\zeta \right| \\
&\leq \int_\gamma |\varphi(\zeta) - \varphi_n(\zeta)|\,|\mathrm{d}\zeta| \\
&< \int_\gamma \frac{\varepsilon}{\text{length}(\gamma)}\,|\mathrm{d}\zeta| \\
&= \frac{\varepsilon}{\text{length}(\gamma)} \int_\gamma |\mathrm{d}\zeta| \\
&= \varepsilon.
\end{aligned}$$

2. The First Application: Higher Derivatives

Let Ω be a region in $\mathbb{C}$. Let $\gamma : I \longrightarrow \Omega$ be a simple closed curve in Ω, traversed counterclockwise. Again the symbol γ will also denote the trace of the curve. Let $f : \Omega \longrightarrow \mathbb{C}$ be a function. Suppose that

- f is continuous on γ.
- For some positive integer k, the $(k-1)$st derivative $f^{(k-1)}$ exists inside γ and has the integral representations

$$\frac{f^{(k-1)}(z)}{(k-1)!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^k}.$$

In particular, the case of $k = 1$ is Cauchy's integral formula, a quick consequence of Cauchy's Theorem if f is already known to be differentiable. But the assumptions being made here when $k = 1$ do *not* include the existence of f'. The point is that the argument to follow will use the integral representation of the $(k-1)$st derivative to *show* that the kth derivative exists and has the analogous integral representation. By induction, it follows that all derivatives of f exist inside γ as soon as f itself is known to be continuous on γ and to have integral representation inside γ. Since these conditions follow when f is known to be once-differentiable, this proves that one complex derivative, not even known to be continuous, implies infinitely many.

Fix a generic point z inside γ. Let B be a closed ball about z entirely inside γ. Let k be a positive integer. Define a function

$$\varphi^{(k)} : B \times \gamma \longrightarrow \mathbb{C}$$

where

$$\varphi^{(k)}(z', \zeta) = \begin{cases} f(\zeta) \cdot \left(\dfrac{\frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k}}{z' - z} \right) & \text{if } z' \neq z, \\ f(\zeta) \cdot \dfrac{k}{(\zeta - z)^{k+1}} & \text{if } z' = z. \end{cases}$$

As shown in an earlier writeup, $\varphi^{(k)}$ is continuous, and therefore uniformly continuous, so that in particular, $\varphi^{(k)}(z', \zeta)$ is within any prescribed closeness to $\varphi(z, \zeta)$ simultaneously for all ζ if z' is close enough to z.

Take a sequence $\{z'_n\}$ in B converging to z. Define the corresponding sequence of functions of one variable,

$$\{\varphi_n^{(k)}\} : \gamma \longrightarrow \mathbb{C}, \quad \varphi_n^{(k)}(\zeta) = \varphi^{(k)}(z'_n, \zeta), \ n = 1, 2, 3, \ldots,$$

and the corresponding limit function (with a slight abuse of notation),

$$\varphi^{(k)} : \gamma \longrightarrow \mathbb{C}, \quad \varphi^{(k)}(\zeta) = \varphi^{(k)}(z, \zeta).$$

The sequence $\{\varphi_n^{(k)}\}$ converges uniformly to $\varphi^{(k)}$. So compute, using the lemma at the third step, that

$$\begin{aligned} \frac{1}{k!} \lim_{n\to\infty} \frac{f^{(k-1)}(z'_n) - f^{(k-1)}(z)}{z'_n - z} &= \frac{1}{k} \lim_{n\to\infty} \frac{\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z'_n)^k} - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^k}}{z'_n - z} \\ &= \frac{1}{k} \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \varphi_n^{(k)}(\zeta)\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\zeta)^{(k)}}{k}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{k+1}}. \end{aligned}$$

Since this calculation holds for every sequence $\{z'_n\}$ in B that converges to z, it shows that $f^{(k)}(z)$ exists and has integral representation

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{k+1}}.$$

At least in the case that γ is piecewise $\mathcal{C}^1$, to produce the same result using the Dominated Convergence Theorem rather than our Uniform Convergence Lemma, we quote the fact that a continuous function on a compact set is bounded, rather than the fact that a continuous function on a compact set is uniformly continuous. Here the function is $\varphi^{(k)}(z', \zeta) : B \times \gamma \longrightarrow \mathbb{C}$. Because it is continuous, the sequence $\{\varphi_n^{(k)}(\zeta)\}$ above converges pointwise to $\varphi^{(k)}(\zeta)$, and because it is bounded, some constant function bounds all functions in the sequence. This is enough for the DCT, because a constant function is integrable over a curve of finite length. The gain in ease here, and the gain in practice at reaching for the best tool to address a problem, need to be balanced against the investment of really understanding the DCT.

3. The Second Application: Power Series Representation

Recall the environment where

- Ω is a region in $\mathbb{C}$,
- $f : \Omega \longrightarrow \mathbb{C}$ is a differentiable function,
- γ is a circle in Ω such that Ω contains all of its interior,
- R is the radius of γ, a is the centerpoint of γ, and z is any point interior to γ.

We defined a sequence of functions

$$\{\varphi_n\} : \gamma \longrightarrow \mathbb{C}, \quad \varphi_n(\zeta) = f(\zeta) \sum_{k=0}^{n} \frac{(z-a)^k}{(\zeta - a)^{k+1}}, \; n = 1, 2, 3, \dots,$$

and then their pointwise limit function,

$$\varphi : \gamma \longrightarrow \mathbb{C}, \quad \varphi(\zeta) = f(\zeta) \sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta - a)^{k+1}}.$$

It follows from the integral representation of f that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - a) - (z - a)} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - a)\left(1 - \frac{z-a}{\zeta - a}\right)}, \end{aligned}$$

so that by the geometric series formula, the calculation continues

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma f(\zeta) \sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta - a)^{k+1}}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \varphi(\zeta)\,\mathrm{d}\zeta. \end{aligned}$$

The sequence $\{\varphi_n\}$ converges to φ uniformly on γ, so by the lemma,

$$\begin{aligned} f(z) &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \varphi_n(\zeta)\,\mathrm{d}\zeta \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma f(\zeta) \sum_{k=0}^{n} \frac{(z-a)^k}{(\zeta - a)^{k+1}}\,\mathrm{d}\zeta. \end{aligned}$$

The finite sum and the powers of $z - a$ pass through the integral, and then the integral representation of the derivatives of f gives the desired power series representation of f,

$$
\begin{aligned}
f(z) &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - a)^{k+1}} (z-a)^k \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (z-a)^k \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k.
\end{aligned}
$$