PASSING LIMITS THROUGH INTEGRALS

1. A General Lemma

Let Ω be a region in $\mathbf{C}$, and let $\gamma : I \longrightarrow \Omega$ be a rectifiable curve. By a small abuse of notation, the symbol γ will also denote the trace of the curve. Let

$$\{\varphi_n\} : \gamma \longrightarrow \mathbf{C}$$

be a sequence of integrable functions converging uniformly to an integrable function

$$\varphi : \gamma \longrightarrow \mathbf{C}.$$

(For example, if each φ_n is continuous then it is integrable, and the uniform convergence then guarantees that φ is continuous and hence integrable as well.) Then

$$\lim_{n\to\infty} \int_\gamma \varphi_n(\zeta)\, d\zeta = \int_\gamma \varphi(\zeta)\, d\zeta.$$

To prove this, let $\varepsilon > 0$ be given. There exists a starting index n_0 such that

$$n \geq n_0 \implies |\varphi(\zeta) - \varphi_n(\zeta)| < \varepsilon/\text{length}(\gamma) \text{ for all } \zeta \in \gamma.$$

It follows that for all $n \geq n_0$,

$$\begin{aligned}
\left| \int_\gamma \varphi(\zeta)\, d\zeta - \int_\gamma \varphi_n(\zeta)\, d\zeta \right| &= \left| \int_\gamma (\varphi(\zeta) - \varphi_n(\zeta))\, d\zeta \right| \\
&\leq \int_\gamma |\varphi(\zeta) - \varphi_n(\zeta)|\, |d\zeta| \\
&\leq \int_\gamma \frac{\varepsilon}{\text{length}(\gamma)}\, |d\zeta| \\
&\leq \frac{\varepsilon}{\text{length}(\gamma)} \int_\gamma |d\zeta| \\
&= \varepsilon.
\end{aligned}$$

2. The First Application: Higher Derivatives

Let Ω be a region in $\mathbf{C}$. Let $\gamma : I \longrightarrow \Omega$ be a simple closed curve in Ω, traversed counterclockwise. Again the symbol γ will also denote the trace of the curve. Let $f : \Omega \longrightarrow \mathbf{C}$ be a function. Suppose that

- f is continuous on γ.
- For some positive integer k, the $(k-1)$st derivative $f^{(k-1)}$ exists inside γ and has the integral representations

$$\frac{f^{(k-1)}(z)}{(k-1)!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^k}.$$

In particular, the case of $k = 1$ is Cauchy's integral formula, a quick consequence of Cauchy's Theorem if f is already known to be differentiable. But the assumptions being made here when $k = 1$ do *not* include the existence of f'. The point is that the argument to follow will use the integral representation of the $(k-1)$st derivative to *show* that the kth derivative exists and has the analogous integral representation. By induction, it follows that all derivatives of f exist inside γ as soon as f itself is known to be continuous on γ and to have integral representation inside γ. Since these conditions follow when f is known to be once-differentiable, this proves that one complex derivative, not even known to be continuous, implies infinitely many.

Fix a generic point z inside γ. Let B be a closed ball about z entirely inside γ. Define a function

$$\varphi : B \times \gamma \longrightarrow \mathbf{C}$$

where

$$\varphi(z', \zeta) = \begin{cases} f(\zeta) \cdot \left(\dfrac{\frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k}}{z' - z} \right) & \text{if } z' \neq z, \\ f(\zeta) \cdot \dfrac{k}{(\zeta - z)^{k+1}} & \text{if } z' = z. \end{cases}$$

Similarly to the example in an earlier piece, φ is continuous because by definition,

$$\lim_{z' \to z} \left(\frac{\frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k}}{z' - z} \right) = \frac{\partial}{\partial z} \left(\frac{1}{(\zeta - z)^k} \right) = \frac{k}{(\zeta - z)^{k+1}}.$$

It follows that φ is uniformly continuous on its domain, and in particular that $\varphi(z', \zeta)$ is within any prescribed closeness to $\varphi(z, \zeta)$ simultaneously for all ζ if z' is close enough to z.

Take a sequence $\{z'_n\}$ in B converging to z. Define the corresponding sequence of functions of one variable,

$$\{\varphi_n\} : \gamma \longrightarrow \mathbf{C}, \quad \varphi_n(\zeta) = \varphi(z'_n, \zeta), \;\; n = 1, 2, 3, \ldots,$$

and the corresponding limit function (with a slight abuse of notation),

$$\varphi : \gamma \longrightarrow \mathbf{C}, \quad \varphi(\zeta) = \varphi(z, \zeta).$$

Then the sequence $\{\varphi_n\}$ converges uniformly to φ. So compute, using the lemma at the third step, that

$$\begin{aligned} \frac{1}{k!} \lim_{n \to \infty} \frac{f^{(k-1)}(z'_n) - f^{(k-1)}(z)}{z'_n - z} &= \frac{1}{k} \lim_{n \to \infty} \frac{\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z'_n)^k} - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^k}}{z'_n - z} \\ &= \frac{1}{k} \lim_{n \to \infty} \frac{1}{2\pi i} \int_\gamma \varphi_n(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\zeta)}{k}\, d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}}. \end{aligned}$$

Since this calculation holds for every sequence $\{z'_n\}$ in B that converges to z, it shows that $f^{(k)}(z)$ exists and has integral representation

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}}.$$

3. The Second Application: Power Series Representation

Recall the environment where

- Ω is a region in $\mathbf{C}$,
- $f : \Omega \longrightarrow \mathbf{C}$ is a differentiable function,
- γ is a circle in Ω such that Ω contains all of its interior,
- R is the radius of γ, a is the centerpoint of γ, and z is any point interior to γ.

We defined a sequence of functions

$$\{\varphi_n\} : \gamma \longrightarrow \mathbf{C}, \quad \varphi_n(\zeta) = f(\zeta) \sum_{j=0}^{n} \frac{(z-a)^j}{(\zeta-a)^{j+1}}, \ n = 1, 2, 3, \dots,$$

and then their pointwise limit function,

$$\varphi : \gamma \longrightarrow \mathbf{C}, \quad \varphi(\zeta) = f(\zeta) \sum_{j=0}^{\infty} \frac{(z-a)^j}{(\zeta-a)^{j+1}}.$$

It follows from the integral representation of f that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta-a) - (z-a)} \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta-a)\left(1 - \frac{z-a}{\zeta-a}\right)}, \end{aligned}$$

so that by the geometric series formula, the calculation continues

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_\gamma f(\zeta) \sum_{j=0}^{\infty} \frac{(z-a)^j}{(\zeta-a)^{j+1}}\, d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \varphi(\zeta)\, d\zeta. \end{aligned}$$

The sequence $\{\varphi_n\}$ converges to φ uniformly on γ, so by the lemma,

$$\begin{aligned} f(z) &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \varphi_n(\zeta)\, d\zeta \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma f(\zeta) \sum_{j=0}^{n} \frac{(z-a)^j}{(\zeta-a)^{j+1}}\, d\zeta. \end{aligned}$$

The finite sum and the powers of $z-a$ pass through the integral, and then the integral representation of the derivatives of f gives the desired power series representation of f,

$$\begin{aligned}f(z) &= \lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)\,d\zeta}{(\zeta-a)^{j+1}}(z-a)^j\\ &= \lim_{n\to\infty}\sum_{j=0}^{n}\frac{f^{(j)}(a)}{j!}(z-a)^j\\ &= \sum_{j=0}^{\infty}\frac{f^{(j)}(a)}{j!}(z-a)^j.\end{aligned}$$