PASSING LIMITS THROUGH INTEGRALS

1. A General Lemma

Let $\Omega$ be a region in $\mathbb{C}$, and let $\gamma : I \to \Omega$ be a rectifiable curve. By a small abuse of notation, the symbol $\gamma$ will also denote the trace of the curve. Let

$$\{\varphi_n\} : \gamma \to \mathbb{C}$$

be a sequence of integrable functions converging uniformly to an integrable function $\varphi : \gamma \to \mathbb{C}$.

For example, if each $\varphi_n$ is continuous then it is integrable, and the uniform convergence then guarantees that $\varphi$ is continuous and hence integrable as well. Then

$$\lim_{n \to \infty} \int_{\gamma} \varphi_n(\zeta) \, d\zeta = \int_{\gamma} \varphi(\zeta) \, d\zeta.$$

To prove this, let $\varepsilon > 0$ be given. We may assume that $\gamma$ has positive length. There exists a starting index $n_0$ such that

$$n \geq n_0 \implies |\varphi(\zeta) - \varphi_n(\zeta)| < \frac{\varepsilon}{\text{length}(\gamma)}$$

for all $\zeta \in \gamma$.

It follows that for all $n \geq n_0$,

$$\left| \int_{\gamma} \varphi(\zeta) \, d\zeta - \int_{\gamma} \varphi_n(\zeta) \, d\zeta \right| = \left| \int_{\gamma} (\varphi(\zeta) - \varphi_n(\zeta)) \, d\zeta \right|$$

$$\leq \int_{\gamma} |\varphi(\zeta) - \varphi_n(\zeta)| \, d\zeta$$

$$< \int_{\gamma} \frac{\varepsilon}{\text{length}(\gamma)} \, d\zeta$$

$$= \frac{\varepsilon}{\text{length}(\gamma)} \int_{\gamma} |d\zeta|$$

$$= \varepsilon.$$

2. The First Application: Higher Derivatives

Let $\Omega$ be a region in $\mathbb{C}$. Let $\gamma : I \to \Omega$ be a simple closed curve in $\Omega$, traversed counterclockwise. Again the symbol $\gamma$ will also denote the trace of the curve. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that

- $f$ is continuous on $\gamma$,
- For some positive integer $k$, the $(k-1)$st derivative $f^{(k-1)}$ exists inside $\gamma$ and has the integral representations

$$f^{(k-1)}(z) = \frac{1}{(k-1)!} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta - z)^k}.$$
In particular, the case of \( k = 1 \) is Cauchy’s integral formula, a quick consequence of Cauchy’s Theorem if \( f \) is already known to be differentiable. But the assumptions being made here when \( k = 1 \) do not include the existence of \( f' \). The point is that the argument to follow will use the integral representation of the \((k - 1)\)st derivative to show that the \( k \)th derivative exists and has the analogous integral representation. By induction, it follows that all derivatives of \( f \) exist inside \( \gamma \) as soon as \( f \) itself is known to be continuous on \( \gamma \) and to have integral representation inside \( \gamma \). Since these conditions follow when \( f \) is known to be once-differentiable, this proves that one complex derivative, not even known to be continuous, implies infinitely many.

Fix a generic point \( z \) inside \( \gamma \). Let \( B \) be a closed ball about \( z \) entirely inside \( \gamma \). Let \( k \) be a positive integer. Define a function

\[
\varphi^{(k)} : B \times \gamma \rightarrow \mathbb{C}
\]

where

\[
\varphi^{(k)}(z', \zeta) = \begin{cases} 
  f(\zeta) \cdot \left( \frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k} \right) & \text{if } z' \neq z, \\
  f(\zeta) \cdot \frac{k}{(\zeta - z)^{k+1}} & \text{if } z' = z.
\end{cases}
\]

As shown in an earlier writeup, \( \varphi^{(k)} \) is continuous, and therefore uniformly continuous, so that in particular, \( \varphi^{(k)}(z', \zeta) \) is within any prescribed closeness to \( \varphi(z, \zeta) \) simultaneously for all \( \zeta \) if \( z' \) is close enough to \( z \).

Take a sequence \( \{z'_n\} \) in \( B \) converging to \( z \). Define the corresponding sequence of functions of one variable,

\[
\{\varphi^{(k)}_n\} : \gamma \rightarrow \mathbb{C}, \quad \varphi^{(k)}_n(\zeta) = \varphi^{(k)}(z'_n, \zeta), \quad n = 1, 2, 3, \ldots,
\]

and the corresponding limit function (with a slight abuse of notation),

\[
\varphi^{(k)} : \gamma \rightarrow \mathbb{C}, \quad \varphi^{(k)}(\zeta) = \varphi^{(k)}(z, \zeta).
\]

The sequence \( \{\varphi^{(k)}_n\} \) converges uniformly to \( \varphi^{(k)} \). So compute, using the lemma at the third step, that

\[
\frac{1}{k!} \lim_{n \to \infty} \frac{f^{(k-1)}(z'_n) - f^{(k-1)}(z)}{z'_n - z} = \frac{1}{k!} \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z'_n)^k} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^k}
\]

\[
= \frac{1}{k!} \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \varphi^{(k)}_n(\zeta) d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi^{(k)}(\zeta)}{k} d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}.
\]

Since this calculation holds for every sequence \( \{z'_n\} \) in \( B \) that converges to \( z \), it shows that \( f^{(k)}(z) \) exists and has integral representation

\[
\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}.
\]

At least in the case that \( \gamma \) is piecewise \( C^1 \), to produce the same result using the Dominated Convergence Theorem rather than our Uniform Convergence Lemma, we quote the fact that a continuous function on a compact set is bounded, rather
than the fact that a continuous function on a compact set is uniformly continuous. Here the function is \( \varphi^{(k)}(z', \zeta) : B \times \gamma \to \mathbb{C} \). Because it is continuous, the sequence \( \{ \varphi^{(k)}_{n}(\zeta) \} \) above converges pointwise to \( \varphi^{(k)}(\zeta) \), and because it is bounded, some constant function bounds all functions in the sequence. This is enough for the DCT, because a constant function is integrable over a curve of finite length. The gain in ease here, and the gain in practice at reaching for the best tool to address a problem, need to be balanced against the investment of really understanding the DCT.

3. The Second Application: Power Series Representation

Recall the environment where

- \( \Omega \) is a region in \( \mathbb{C} \),
- \( f : \Omega \to \mathbb{C} \) is a differentiable function,
- \( \gamma \) is a circle in \( \Omega \) such that \( \Omega \) contains all of its interior,
- \( R \) is the radius of \( \gamma \), \( a \) is the centerpoint of \( \gamma \), and \( z \) is any point interior to \( \gamma \).

We defined a sequence of functions

\[
\{ \varphi_{n} \} : \gamma \to \mathbb{C}, \quad \varphi_{n}(\zeta) = f(\zeta) \sum_{k=0}^{n} \frac{(z-a)^{k}}{(\zeta-a)^{k+1}}, \quad n = 1, 2, 3, \ldots,
\]

and then their pointwise limit function,

\[
\varphi : \gamma \to \mathbb{C}, \quad \varphi(\zeta) = f(\zeta) \sum_{k=0}^{\infty} \frac{(z-a)^{k}}{(\zeta-a)^{k+1}}.
\]

It follows from the integral representation of \( f \) that

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{\zeta - z}
= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta-a) - (z-a)}
= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta-a) \left(1 - \frac{z-a}{\zeta-a}\right)},
\]

so that by the geometric series formula, the calculation continues

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \sum_{k=0}^{\infty} \frac{(z-a)^{k}}{(\zeta-a)^{k+1}} \, d\zeta
= \frac{1}{2\pi i} \int_{\gamma} \varphi(\zeta) \, d\zeta.
\]

The sequence \( \{ \varphi_{n} \} \) converges to \( \varphi \) uniformly on \( \gamma \), so by the lemma,

\[
f(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \varphi_{n}(\zeta) \, d\zeta
= \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \sum_{k=0}^{n} \frac{(z-a)^{k}}{(\zeta-a)^{k+1}} \, d\zeta.
\]
The finite sum and the powers of $z - a$ pass through the integral, and then the integral representation of the derivatives of $f$ gives the desired power series representation of $f$,

$$f(z) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta - a)^{k+1}} (z - a)^k$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (z - a)^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k.$$