PASSING LIMITS THROUGH INTEGRALS

1. A General Lemma

Let $\Omega$ be a region in $\mathbb{C}$, and let $\gamma : I \to \Omega$ be a rectifiable curve. By a small abuse of notation, the symbol $\gamma$ will also denote the trace of the curve. Let $\{\phi_n\} : \gamma \to \mathbb{C}$ be a sequence of integrable functions converging uniformly to an integrable function $\phi : \gamma \to \mathbb{C}$.

For example, if each $\phi_n$ is continuous then it is integrable, and the uniform convergence then guarantees that $\phi$ is continuous and hence integrable as well. Then

$$\lim_{n \to \infty} \int_{\gamma} \phi_n(\zeta) \, d\zeta = \int_{\gamma} \phi(\zeta) \, d\zeta.$$ 

To prove this, let $\varepsilon > 0$ be given. We may assume that $\gamma$ has positive length. There exists a starting index $n_0$ such that

$$n \geq n_0 \implies |\phi(\zeta) - \phi_n(\zeta)| < \frac{\varepsilon}{\text{length}(\gamma)}$$

for all $\zeta \in \gamma$.

It follows that for all $n \geq n_0$,

$$\left| \int_{\gamma} \phi(\zeta) \, d\zeta - \int_{\gamma} \phi_n(\zeta) \, d\zeta \right| = \left| \int_{\gamma} (\phi(\zeta) - \phi_n(\zeta)) \, d\zeta \right|$$

$$\leq \int_{\gamma} |\phi(\zeta) - \phi_n(\zeta)| \, |d\zeta|$$

$$< \int_{\gamma} \frac{\varepsilon}{\text{length}(\gamma)} |d\zeta|$$

$$= \frac{\varepsilon}{\text{length}(\gamma)} \int_{\gamma} |d\zeta|$$

$$= \varepsilon.$$ 

2. The First Application: Higher Derivatives

Let $\Omega$ be a region in $\mathbb{C}$. Let $\gamma : I \to \Omega$ be a simple closed curve in $\Omega$, traversed counterclockwise. Again the symbol $\gamma$ will also denote the trace of the curve. Let $f : \Omega \to \mathbb{C}$ be a function. Suppose that

- $f$ is continuous on $\gamma$,
- For some positive integer $k$, the $(k-1)$st derivative $f^{(k-1)}$ exists inside $\gamma$ and has the integral representations

$$\frac{f^{(k-1)}(z)}{(k-1)!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^k} \, d\zeta.$$
In particular, the case of $k = 1$ is Cauchy’s integral formula, a quick consequence of Cauchy’s Theorem if $f$ is already known to be differentiable. But the assumptions being made here when $k = 1$ do not include the existence of $f'$. The point is that the argument to follow will use the integral representation of the $(k-1)$st derivative to show that the $k$th derivative exists and has the analogous integral representation. By induction, it follows that all derivatives of $f$ exist inside $\gamma$ as soon as $f$ itself is known to be continuous on $\gamma$ and to have integral representation inside $\gamma$. Since these conditions follow when $f$ is known to be once-differentiable, this proves that one complex derivative, not even known to be continuous, implies infinitely many.

Fix a generic point $z$ inside $\gamma$. Let $B$ be a closed ball about $z$ entirely inside $\gamma$. Let $k$ be a positive integer. Define a function

$$\varphi^{(k)} : B \times \gamma \rightarrow \mathbb{C}$$

where

$$\varphi^{(k)}(z', \zeta) = \begin{cases} f(\zeta) \cdot \left( \frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k} \right) & \text{if } z' \neq z, \\ f(\zeta) \cdot \frac{k}{(\zeta - z)^{k+1}} & \text{if } z' = z. \end{cases}$$

As shown in an earlier writeup, $\varphi^{(k)}$ is continuous, and therefore uniformly continuous, so that in particular, $\varphi^{(k)}(z', \zeta)$ is within any prescribed closeness to $\varphi(z, \zeta)$ simultaneously for all $\zeta$ if $z'$ is close enough to $z$.

Take a sequence \{${z'_n}$\} in $B$ converging to $z$. Define the corresponding sequence of functions of one variable,

$$\{\varphi^{(k)}_n\} : \gamma \rightarrow \mathbb{C}, \quad \varphi^{(k)}_n(\zeta) = \varphi(z'_n, \zeta), \ n = 1, 2, 3, \ldots,$$

and the corresponding limit function (with a slight abuse of notation),

$$\varphi^{(k)} : \gamma \rightarrow \mathbb{C}, \quad \varphi^{(k)}(\zeta) = \varphi^{(k)}(z, \zeta).$$

Then the sequence \{${\varphi^{(k)}_n}$\} converges uniformly to $\varphi^{(k)}$. So compute, using the lemma at the third step, that

$$\frac{1}{k!} \lim_{n \to \infty} \frac{f^{(k-1)}(z'_n) - f^{(k-1)}(z)}{z'_n - z} = \frac{1}{k!} \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z'_n)^k} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^k}$$

$$= \frac{1}{k!} \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi^{(k)}_n(\zeta) d\zeta}{k}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{k}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}.$$ 

Since this calculation holds for every sequence \{${z'_n}$\} in $B$ that converges to $z$, it shows that $f^{(k)}(z)$ exists and has integral representation

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}.$$ 

At least in the case that $\gamma$ is piecewise $C^1$, to produce the same result using the Dominated Convergence Theorem rather than our Uniform Convergence Lemma, we quote the fact that a continuous function on a compact set is bounded, rather
than the fact that a continuous function on a compact set is uniformly continuous. Here the function is $\varphi^{(k)}(z', \zeta) : B \times \gamma \rightarrow \mathbb{C}$. Because it is continuous, the sequence $\{\varphi^{(k)}_n(\zeta)\}$ above converges pointwise to $\varphi^{(k)}(\zeta)$, and because it is bounded, some constant function bounds all functions in the sequence. This is enough for the DCT, because a constant function is integrable over a curve of finite length. The gain in ease here, and the gain in practice at reaching for the best tool to address a problem, need to be balanced against the investment of really understanding the DCT.

3. The Second Application: Power Series Representation

Recall the environment where

- $\Omega$ is a region in $\mathbb{C}$,
- $f : \Omega \rightarrow \mathbb{C}$ is a differentiable function,
- $\gamma$ is a circle in $\Omega$ such that $\Omega$ contains all of its interior,
- $R$ is the radius of $\gamma$, $a$ is the centerpoint of $\gamma$, and $z$ is any point interior to $\gamma$.

We defined a sequence of functions

$$\{\varphi_n\} : \gamma \rightarrow \mathbb{C}, \quad \varphi_n(\zeta) = f(\zeta) \sum_{k=0}^{n} \frac{(z-a)^k}{(\zeta-a)^{k+1}}, \quad n = 1, 2, 3, \ldots,$$

and then their pointwise limit function,

$$\varphi : \gamma \rightarrow \mathbb{C}, \quad \varphi(\zeta) = \lim_{n \to \infty} f(\zeta) \sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta-a)^{k+1}}.$$

It follows from the integral representation of $f$ that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta-a) - (z-a)}$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta-a) \left(1 - \frac{z-a}{\zeta-a}\right)}.$$ 

so that by the geometric series formula, the calculation continues

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \sum_{k=0}^{\infty} \frac{(z-a)^k}{(\zeta-a)^{k+1}} d\zeta$$

$$= \frac{1}{2\pi i} \int_{\gamma} \varphi(\zeta) d\zeta.$$

The sequence $\{\varphi_n\}$ converges to $\varphi$ uniformly on $\gamma$, so by the lemma,

$$f(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \varphi_n(\zeta) d\zeta$$

$$= \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \sum_{k=0}^{n} \frac{(z-a)^k}{(\zeta-a)^{k+1}} d\zeta.$$
The finite sum and the powers of $z - a$ pass through the integral, and then the integral representation of the derivatives of $f$ gives the desired power series representation of $f$,

$$f(z) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - a)^{k+1}} (z - a)^k$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (z - a)^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k.$$