1. A General Lemma

Let $\Omega$ be a region in $\mathbb{C}$, and let $\gamma : I \rightarrow \Omega$ be a rectifiable curve. By a small abuse of notation, the symbol $\gamma$ will also denote the trace of the curve. Let

$$\{\varphi_n\} : \gamma \rightarrow \mathbb{C}$$

be a sequence of integrable functions converging uniformly to an integrable function

$$\varphi : \gamma \rightarrow \mathbb{C}.$$ 

(For example, if each $\varphi_n$ is continuous then it is integrable, and the uniform convergence then guarantees that $\varphi$ is continuous and hence integrable as well.) Then

$$\lim_{n \to \infty} \int_{\gamma} \varphi_n(\zeta) \, d\zeta = \int_{\gamma} \varphi(\zeta) \, d\zeta.$$ 

To prove this, let $\varepsilon > 0$ be given. There exists a starting index $n_0$ such that

$$n \geq n_0 \implies |\varphi(\zeta) - \varphi_n(\zeta)| < \varepsilon / \text{length}(\gamma)$$

for all $\zeta \in \gamma$. It follows that for all $n \geq n_0$,

$$\left| \int_{\gamma} \varphi(\zeta) \, d\zeta - \int_{\gamma} \varphi_n(\zeta) \, d\zeta \right| = \left| \int_{\gamma} (\varphi(\zeta) - \varphi_n(\zeta)) \, d\zeta \right|$$

$$\leq \int_{\gamma} |\varphi(\zeta) - \varphi_n(\zeta)| \, |d\zeta|$$

$$\leq \int_{\gamma} \frac{\varepsilon}{\text{length}(\gamma)} \, |d\zeta|$$

$$\leq \frac{\varepsilon}{\text{length}(\gamma)} \int_{\gamma} |d\zeta|$$

$$= \varepsilon.$$

2. The First Application: Higher Derivatives

Let $\Omega$ be a region in $\mathbb{C}$. Let $\gamma : I \rightarrow \Omega$ be a simple closed curve in $\Omega$, traversed counterclockwise. Again the symbol $\gamma$ will also denote the trace of the curve. Let $f : \Omega \rightarrow \mathbb{C}$ be a function. Suppose that

- $f$ is continuous on $\gamma$.
- For some positive integer $k$, the $(k - 1)$st derivative $f^{(k-1)}$ exists inside $\gamma$ and has the integral representations

$$\frac{f^{(k-1)}(z)}{(k-1)!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta - z)^k}.$$
In particular, the case of $k = 1$ is Cauchy’s integral formula, a quick consequence of Cauchy’s Theorem if $f$ is already known to be differentiable. But the assumptions being made here when $k = 1$ do not include the existence of $f'$. The point is that the argument to follow will use the integral representation of the $(k - 1)$st derivative to show that the $k$th derivative exists and has the analogous integral representation. By induction, it follows that all derivatives of $f$ exist inside $\gamma$ as soon as $f$ itself is known to be continuous on $\gamma$ and to have integral representation inside $\gamma$. Since these conditions follow when $f$ is known to be once-differentiable, this proves that one complex derivative, not even known to be continuous, implies infinitely many.

Fix a generic point $z$ inside $\gamma$. Let $B$ be a closed ball about $z$ entirely inside $\gamma$. Define a function

$$\varphi : B \times \gamma \rightarrow \mathbb{C}$$

where

$$\varphi(z', \zeta) = \begin{cases} f(\zeta) \cdot \left( \frac{1}{(\zeta - z)^k} - \frac{1}{(\zeta - z)^{k+1}} \right) & \text{if } z' \neq z, \\ f(\zeta) & \text{if } z' = z. \end{cases}$$

Similarly to the example in an earlier piece, $\varphi$ is continuous because by definition,

$$\lim_{z' \to z} \left( \frac{1}{(\zeta - z)^k} - \frac{1}{(\zeta - z)^{k+1}} \right) = \frac{\partial}{\partial z} \left( \frac{1}{(\zeta - z)^k} \right) = \frac{k}{(\zeta - z)^{k+1}}.$$ \hfill (1)

It follows that $\varphi$ is uniformly continuous on its domain, and in particular that $\varphi(z', \zeta)$ is within any prescribed closeness to $\varphi(z, \zeta)$ simultaneously for all $\zeta$ if $z'$ is close enough to $z$.

Take a sequence $\{z'_n\}$ in $B$ converging to $z$. Define the corresponding sequence of functions of one variable,

$$\{\varphi_n\} : \gamma \rightarrow \mathbb{C}, \quad \varphi_n(\zeta) = \varphi(z'_n, \zeta), \quad n = 1, 2, 3, \ldots,$$

and the corresponding limit function (with a slight abuse of notation),

$$\varphi : \gamma \rightarrow \mathbb{C}, \quad \varphi(\zeta) = \varphi(z, \zeta).$$

Then the sequence $\{\varphi_n\}$ converges uniformly to $\varphi$. So compute, using the lemma at the third step, that

$$\frac{1}{k!} \lim_{n \to \infty} \frac{f^{(k-1)}(z'_n) - f^{(k-1)}(z)}{z'_n - z} = \frac{1}{k!} \lim_{n \to \infty} \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z'_n)^k} - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}$$

$$= \frac{1}{k!} \lim_{n \to \infty} \frac{1}{2\pi i} \int_\gamma \varphi_n(\zeta) d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\zeta)}{k} d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}.$$

Since this calculation holds for every sequence $\{z'_n\}$ in $B$ that converges to $z$, it shows that $f^{(k)}(z)$ exists and has integral representation

$$\frac{f^{(k)}(z)}{k!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z)^{k+1}}.$$

3. THE SECOND APPLICATION: POWER SERIES REPRESENTATION

Recall the environment where

- \( \Omega \) is a region in \( \mathbb{C} \),
- \( f : \Omega \rightarrow \mathbb{C} \) is a differentiable function,
- \( \gamma \) is a circle in \( \Omega \) such that \( \Omega \) contains all of its interior,
- \( R \) is the radius of \( \gamma \), \( a \) is the centerpoint of \( \gamma \), and \( z \) is any point interior to \( \gamma \).

We defined a sequence of functions

\[
\{ \varphi_n \} : \gamma \rightarrow \mathbb{C}, \quad \varphi_n(\zeta) = f(\zeta) \sum_{j=0}^{n} \frac{(z-a)^j}{(\zeta-a)^{j+1}}, \quad n = 1, 2, 3, \ldots,
\]

and then their pointwise limit function,

\[
\varphi : \gamma \rightarrow \mathbb{C}, \quad \varphi(\zeta) = f(\zeta) \sum_{j=0}^{\infty} \frac{(z-a)^j}{(\zeta-a)^{j+1}}.
\]

It follows from the integral representation of \( f \) that

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta - a) - (z - a)} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta - a) \left(1 - \frac{z-a}{\zeta-a}\right)},
\]

so that by the geometric series formula, the calculation continues

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \sum_{j=0}^{\infty} \frac{(z-a)^j}{(\zeta-a)^{j+1}} \, d\zeta = \frac{1}{2\pi i} \int_{\gamma} \varphi(\zeta) \, d\zeta.
\]

The sequence \( \{\varphi_n\} \) converges to \( \varphi \) uniformly on \( \gamma \), so by the lemma,

\[
f(z) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \varphi_n(\zeta) \, d\zeta = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \sum_{j=0}^{n} \frac{(z-a)^j}{(\zeta-a)^{j+1}} \, d\zeta.
\]
The finite sum and the powers of $z - a$ pass through the integral, and then the integral representation of the derivatives of $f$ gives the desired power series representation of $f$,

$$f(z) = \lim_{n \to \infty} \sum_{j=0}^{n} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{(\zeta - a)^{j+1}} (z - a)^j$$

$$= \lim_{n \to \infty} \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!} (z - a)^j$$

$$= \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!} (z - a)^j.$$