MATH 311: COMPLEX ANALYSIS — AUTOMORPHISM GROUPS LECTURE

1. Introduction

Rather than study individual examples of conformal mappings one at a time, we now want to study *families* of conformal mappings. Ensembles of conformal mappings naturally carry group structures.

2. Automorphisms of the Plane

The automorphism group of the complex plane is

$$\mathrm{Aut}(\mathbb{C}) = \{\text{analytic bijections } f : \mathbb{C} \longrightarrow \mathbb{C}\}.$$

Any automorphism of the plane must be conformal, for if $f'(z) = 0$ for some z then f takes the value $f(z)$ with multiplicity $n > 1$, and so by the Local Mapping Theorem it is n-to-1 near z, impossible since f is an automorphism.

By a problem on the midterm, we know the form of such automorphisms: they are

$$f(z) = az + b, \quad a, b \in \mathbb{C},\ a \neq 0.$$

This description of such functions one at a time loses track of the group structure. If $f(z) = az + b$ and $g(z) = a'z + b'$ then

$$\begin{aligned}(f \circ g)(z) &= aa'z + (ab' + b),\\ f^{-1}(z) &= a^{-1}z - a^{-1}b.\end{aligned}$$

But these formulas are not very illuminating. For a better picture of the automorphism group, represent each automorphism by a 2-by-2 complex matrix,

$$f(z) = ax + b \quad \longleftrightarrow \quad \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}. \tag{1}$$

Then the matrix calculations

$$\begin{aligned}\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a' & b' \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} aa' & ab' + b \\ 0 & 1 \end{bmatrix},\\ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} &= \begin{bmatrix} a^{-1} & -a^{-1}b \\ 0 & 1 \end{bmatrix}\end{aligned}$$

naturally encode the formulas for composing and inverting automorphisms of the plane. With this in mind, define the *parabolic group* of 2-by-2 complex matrices,

$$P = \left\{\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{C},\ a \neq 0\right\}.$$

Then the correspondence (1) is a natural group isomorphism,

$$\boxed{\mathrm{Aut}(\mathbb{C}) \cong P.}$$

Two subgroups of the parabolic subgroup are its *Levi component*

$$M = \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} : a \in \mathbb{C},\ a \neq 0 \right\},$$

describing the dilations $f(z) = ax$, and its *unipotent radical*

$$N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : b \in \mathbb{C} \right\},$$

describing the translations $f(z) = z + b$.

Proposition 2.1. *The parabolic group takes the form*

$$P = MN = NM.$$

Also, M normalizes N, meaning that

$$m^{-1}nm \in N \quad \textit{for all } m \in M \textit{ and } n \in N.$$

Proof. To establish the first statement, simply compute:

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly for the second statement,

$$\begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix}.$$

□

The geometric content of the proposition's first statement is that any affine map is the composition of a translation and a dilation and is also the composition of a dilation and a translation. The content of the second statement is that a dilation followed by a translation followed by the reciprocal dilation is again a translation. (I do not find this last result quickly obvious geometrically.)

In sum so far, considering the automorphisms of the plane has led us to affine maps and the parabolic group.

3. Automorphisms of the Sphere

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and consider its automorphism group,

$$\mathrm{Aut}(\widehat{\mathbb{C}}) = \{\text{meromorphic bijections } f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}\}.$$

The fact that meromorphic bijections of the Riemann sphere are closed under composition and inversion follows from the fact that meromorphic functions are analytic in local coordinates, and (excepting the constant function ∞) conversely. In any case, the following lemma will soon make the closure properties of automorphisms of the Riemann sphere clear in more concrete terms.

Proposition 3.1. *Let $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be meromorphic. Then f is a rational function,*

$$f(z) = p(z)/q(z),$$

where p and q are polynomials with complex coefficients, and q is not the zero polynomial.

Proof. Since the Riemann sphere is compact, f can have only finitely many poles, for otherwise a sequence of poles would cluster somewhere, giving a nonisolated singularity. Especially, f has only finitely many poles in the plane. Let the poles occur at the points z_1 through z_n with multiplicities e_1 through e_n. Define a polynomial

$$q(z) = \prod_{j=1}^{n} (z - z_j)^{e_j}, \quad z \in \mathbb{C}.$$

(So $q(z)$ is identically 1 if f has no poles in $\mathbb{C}$.) Then the function

$$p : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \quad p(z) = f(z)q(z)$$

has removable singularities at the poles of f in $\mathbb{C}$, i.e., it is entire. So p has a power series representation on all of $\mathbb{C}$. Also, p is meromorphic at ∞ because both f and q are. This forces p to be a polynomial. Since $f = p/q$, the proof is complete. □

It follows from the proposition that the invertible meromorphic functions on $\widehat{\mathbb{C}}$ take the form

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C},$$

since if the numerator or the denominator of f were to have degree greater than 1 then by the standard argument using the Local Mapping Theorem, f would not be bijective. (To analyze the denominator, consider the reciprocal function $1/f$.) On the other hand, unless at least one of the numerator or the denominator of f has degree 1—as compared to being constant—then again f is not bijective. Also, we are assuming that f is expressed in lowest terms, i.e., the numerator is not a scalar multiple of the denominator. This discussion narrows our considerations to functions of the form

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0. \tag{2}$$

Perhaps it deserves explicit mention here that if $c = 0$ then $f(\infty) = \infty$, while if $c \neq 0$ then $f(-d/c) = \infty$ and $f(\infty) = a/c$. We still don't know that these functions are bijections of the Riemann sphere, but we do know that they are its only possible meromorphic bijections.

Introduce the *general linear group* of 2-by-2 complex matrices,

$$\mathrm{GL}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0 \right\}.$$

Then there is a surjective map

$$\mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{Aut}(\widehat{\mathbb{C}}), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto f(z) = \frac{az+b}{cz+d}.$$

(True, we don't yet know that f is an automorphism, but soon we will, and so we temporarily notate it as such to avoid temporary clutter.) One can verify that the map is a homomorphism, i.e., if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto f(z) = \frac{az+b}{cz+d}, \quad \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \longmapsto g(z) = \frac{a'z+b'}{c'z+d'}$$

then the matrix product maps to the corresponding composition. That is, the product is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix},$$

while the corresponding composition is the image of the product,

$$(f \circ g)(z) = \frac{(aa' + bc')z + (ab' + bd')}{(ca' + dc')z + cb' + dd'}.$$

The verification is discussed in a separate writeup, as it deserves a better treatment than it usually receives. Here we now take it as proved.

Thus the map from $\mathrm{GL}_2(\mathbb{C})$ to $\mathrm{Aut}(\widehat{\mathbb{C}})$ is an epimorphism, meaning a surjective homomorphism. This makes it easy to show that all maps of the form (2) are automorphisms. Any such function arises from an invertible matrix, and the inverse matrix gives rise to the inverse function. As automorphisms of a structure that locally looks like the complex plane, all functions (2) are conformal. The calculation

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0$$

verifies this directly so long as the infinite point of the Riemann sphere is not involved, but strictly speaking a complete argument requires special-case calculations to cover the cases where the input to f or the output from f is ∞.

The map from $\mathrm{GL}_2(\mathbb{C})$ to $\mathrm{Aut}(\widehat{\mathbb{C}})$ is not injective because all nonzero scalar multiples of a given matrix are taken to the same automorphism. The kernel of the map (the inputs that it takes to the identity automorphism) is the subgroup of $\mathrm{GL}_2(\mathbb{C})$ consisting of all nonzero scalar multiples of the identity matrix,

$$\mathbb{C}^\times I = \left\{ \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : \lambda \in \mathbb{C},\ \lambda \neq 0 \right\}.$$

And so by the First Isomorphism Theorem of group theory, there is an isomorphism

$$\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times I \xrightarrow{\sim} \mathrm{Aut}(\widehat{\mathbb{C}}), \qquad \left\{ \lambda \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} \longmapsto f(z) = \frac{az + b}{cz + d}.$$

The quotient of $\mathrm{GL}_2(\mathbb{C})$ by the nonzero scalar multiples of the identity matrix is the *projective general linear group* of 2-by-2 complex matrices,

$$\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times I.$$

Next define the *special linear group* of 2-by-2 complex matrices, the elements of the general linear group with determinant 1,

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C}) : ad - bc = 1 \right\}.$$

Introduce the notation

$$G = \mathrm{GL}_2(\mathbb{C}), \quad K = \mathbb{C}^\times I, \quad H = \mathrm{SL}_2(\mathbb{C}),$$

so that $H \cap K = \{\pm I\}$. Then the calculation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\sqrt{ad - bc}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \sqrt{ad - bc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(where $\sqrt{ad-bc}$ denotes either square root of $ad-bc$) shows that $G = HK$. By the Second Isomorphism Theorem of group theory,

$$G/K = HK/K \cong H/(H \cap K),$$

which is to say,

$$\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times I \cong \mathrm{SL}_2(\mathbb{C})/\{\pm I\}.$$

So, letting $\mathrm{PSL}_2(\mathbb{C})$ denote the *projective special linear group* $\mathrm{SL}_2(\mathbb{C})/\{\pm I\}$,

$$\boxed{\mathrm{Aut}(\widehat{\mathbb{C}}) \cong \mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C}).}$$

Note that although $\mathrm{SL}_2(\mathbb{C})$ is very much a proper subgroup of $\mathrm{GL}_2(\mathbb{C})$, the difference between them collapses under projectivizing. Note also that the automorphisms of $\widehat{\mathbb{C}}$ that fix ∞ restrict to automorphisms of $\mathbb{C}$. Correspondingly there is a monomorphism (injective homomorphism) of groups

$$P \longrightarrow \mathrm{PGL}_2(\mathbb{C}), \qquad \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \longmapsto \mathbb{C}^\times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

In sum, considering the automorphisms of the sphere has led us to fractional linear transformations and to the projective general (or special) linear group.

4. Rotations of the Riemann Sphere

The round sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

has its group of rotations, denoted $\mathrm{Rot}(S^2)$. This rotation group is isomorphic to the special orthogonal group of 3-by-3 real matrices

$$\mathrm{Rot}(S^2) \cong \mathrm{SO}_3(\mathbb{R})$$

where the special orthogonal group is defined intrinsically as follows (in which the superscript t denotes the transpose of a matrix):

$$\mathrm{SO}_3(\mathbb{R}) = \{m \in \mathrm{SL}_3(\mathbb{R}) : m^t\, m = I\}.$$

Recall that stereographic projection is a conformal bijection from the round sphere S^2 to the Riemann sphere $\widehat{\mathbb{C}}$. An automorphism of $\widehat{\mathbb{C}}$ that corresponds under stereographic projection to a rotation of S^2 is called a *rotation* of $\widehat{\mathbb{C}}$. The group of rotations of $\widehat{\mathbb{C}}$ is denoted $\mathrm{Rot}(\widehat{\mathbb{C}})$. Thus under stereographic projection,

$$\mathrm{Rot}(S^2) \cong \mathrm{Rot}(\widehat{\mathbb{C}}).$$

The *special unitary* subgroup of $\mathrm{SL}_2(\mathbb{C})$ is defined intrinsically as follows (in which the superscript $*$ denotes the transpose-conjugate of a matrix):

$$\mathrm{SU}_2(\mathbb{C}) = \{m \in \mathrm{SL}_2(\mathbb{C}) : m^* m = I,\ \det m = 1\}.$$

Thus the elements of $\mathrm{SU}_2(\mathbb{C})$ are the 2-by-2 analogues of unit complex numbers. The special unitary group can be described in coordinates,

$$\mathrm{SU}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

This description shows that $\mathrm{SU}_2(\mathbb{C})$ can be viewed as a group structure on the three-dimensional unit sphere $S^3 \subset \mathbb{R}^4$, a compact set, and similarly for $\mathrm{PSU}_2(\mathbb{C})$ and the projective three-sphere, meaning the three-sphere modulo antipodal identification.

Let $\mathrm{PSU}_2(\mathbb{C}) = \mathrm{SU}_2(\mathbb{C})/\{\pm I\}$. A separate writeup establishes the isomorphism

$$\boxed{\mathrm{Rot}(\widehat{\mathbb{C}}) \cong \mathrm{PU}_2(\mathbb{C}) \cong \mathrm{PSU}_2(\mathbb{C}).}$$

(Of course the unitary group $\mathrm{U}_2(\mathbb{C})$ is defined like the special unitary group but without the determinant condition. The isomorphism between the projective unitary and special unitary groups follows from that between the projective general and special linear groups.) A corollary isomorphism is therefore

$$\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{PSU}_2(\mathbb{C}).$$

5. Automorphisms of the Unit Disk

Recall the definition of the special unitary group of 2-by-2 complex matrices intrinsically,

$$\mathrm{SU}_2(\mathbb{C}) = \left\{ m \in \mathrm{SL}_2(\mathbb{C}) : m^* \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \det m = 1 \right\},$$

and in coordinates,

$$\mathrm{SU}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} : a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

Define analogously another special unitary group of 2-by-2 complex matrices,

$$\mathrm{SU}_{1,1}(\mathbb{C}) = \{ m \in \mathrm{SL}_2(\mathbb{C}) : m^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} m = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \det m = 1 \},$$

and in coordinates,

$$\mathrm{SU}_{1,1}(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ \overline{b} & \overline{a} \end{bmatrix} : a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\}.$$

The intrinsic definitions show that $\mathrm{SU}_2(\mathbb{C})$ preserves a geometry with two positive curvatures, whereas $\mathrm{SU}_{1,1}(\mathbb{C})$ preserves a geometry with one positive curvature and one negative curvature, i.e., a hyperbolic geometry. This hyperbolic geometry describes the complex unit disk D,

$$D = \{ z \in \mathbb{C} : |z|^2 < 1 \} = \left\{ z \in \mathbb{C} : \begin{bmatrix} z \\ 1 \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < 0 \right\},$$

and this description of the disk D combines with the intrinsic description of the group $\mathrm{SU}_{1,1}(\mathbb{C})$ to show that the group preserves the disk. Let $\mathrm{PSU}_{1,1}(\mathbb{C}) = \mathrm{SU}_{1,1}(\mathbb{C})/\{\pm I\}$. Then we have shown that

$$\mathrm{PSU}_{1,1}(\mathbb{C}) \subset \mathrm{Aut}(D).$$

That is, we have found *some* automorphisms of the disk. In fact, it will turn out that we have found them all.

(For a coordinate-based argument that $\mathrm{SU}_{1,1}(\mathbb{C})$ preserves D, consider any element m of $\mathrm{SU}_{1,1}(\mathbb{C})$ and any complex number z such that $|z| = 1$, recall that $\overline{z} = z^{-1}$ and compute that consequently

$$|m(z)|^2 = \frac{az + b}{\overline{b}z + \overline{a}} \cdot \frac{\overline{a}z^{-1} + \overline{b}}{bz^{-1} + a} = \frac{az + b}{\overline{b}z + \overline{a}} \cdot \frac{\overline{a} + \overline{b}z}{b + az} = 1.$$

Thus m takes the unit circle to itself, either taking its interior D to itself as well or exchanging D with the exterior of the circle. But compute that

$$|m(0)|^2 = \frac{|b|^2}{|a|^2} < 1 \quad \text{since } |a|^2 = |b|^2 + 1,$$

and so m is an automorphism of D.)

The first step toward seeing that $\mathrm{PSU}_{1,1}(\mathbb{C})$ is all of $\mathrm{Aut}(D)$ is the observation that $\mathrm{PSU}_{1,1}(\mathbb{C})$ acts transitively on D, meaning that the group can move any point of D to any other. It suffices to show that the group can move any point $a \in D$ to the origin. Given such a, consider the matrix

$$m_a = \frac{1}{\sqrt{1-|a|^2}} \begin{bmatrix} 1 & -a \\ -\overline{a} & 1 \end{bmatrix} \in \mathrm{SU}_{1,1}(\mathbb{C}).$$

The corresponding fractional linear transformation is

$$m_a(z) = \frac{z-a}{1-\overline{a}z},$$

and so in particular, $m_a(a) = 0$ as desired. For future reference, observe that $m_a(0) = -a$ and $m_a^{-1} = m_{-a}$. (See figure 1.)

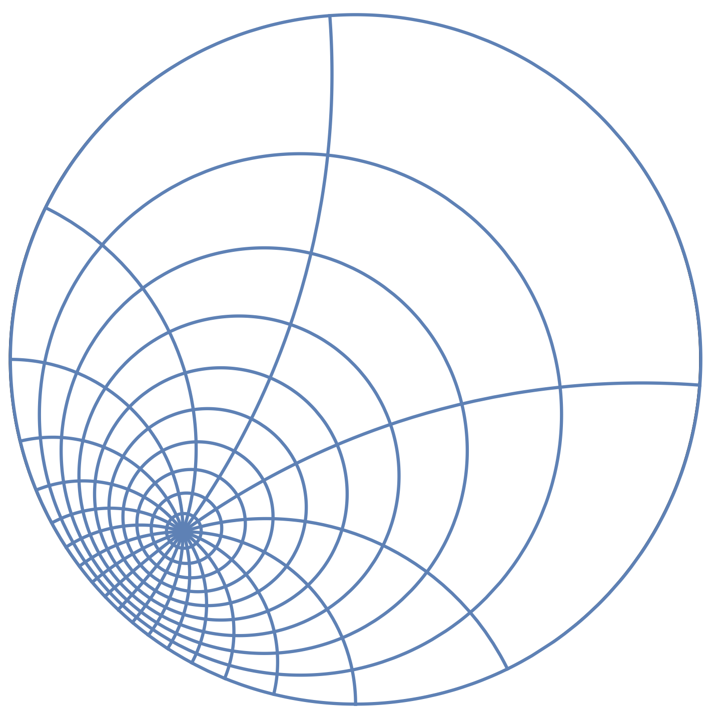

Figure 1. A motion of the disk

So now, given an *arbitrary automorphism* f of D, let a the element of D such that $f(a) = 0$. Then the composition $f \circ m_{-a}$ is an automorphism of D that fixes 0. Such automorphisms must take a surprisingly simple form:

Theorem 5.1 (The Schwarz Lemma). *Let f be an endomorphism of D that fixes 0. Then f is an automorphism if and only if f is a rotation $f(z) = e^{i\theta}z$ for some fixed angle θ.*

Proof. Certainly rotation $f(z) = e^{i\theta}z$ is an automorphism of D. We need to prove the converse.

Given $f : D \longrightarrow D$ with $f(0) = 0$, define a related function

$$g : D - \{0\} \longrightarrow \mathbb{C}, \quad g(z) = f(z)/z.$$

Thus g is analytic on the punctured disk. Furthermore, the singularity of g at 0 is removable since $\lim_{z\to 0} g(z) = f'(0)$. Thus g extends analytically to the disk,

$$g : D \longrightarrow \mathbb{C}, \quad g(z) = \begin{cases} f(z)/z & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0. \end{cases}$$

For any radius r such that $0 < r < 1$ the Maximum Principle gives

$$\sup_{|z|\le r} |g(z)| = \sup_{|z|=r} |g(z)| = \sup_{|z|=r} |f(z)|/r \le 1/r,$$

and so letting $r \to 1^-$ now gives

$$\sup_{z\in D} |g(z)| \le 1.$$

(I find this little point appealing: because r appears in the denominator of the bound on $|g|$, enlarging it gives a *stronger* bound on *more* values of g.) That is, $|g(z)| \le 1$ for all $z \in D$. Furthermore, if $|g(z)| = 1$ for any $z \in D$ then g is the constant function $e^{i\theta}$ for some θ. Returning to the original endomorphism f of D that fixes 0, we have shown that $|f(z)| \le |z|$ on D and that $|f'(0)| \le 1$, and furthermore that if $|f(z)| = |z|$ for some nonzero $z \in D$ or if $|f'(0)| = 1$ then f is the rotation $f(z) = e^{i\theta}z$.

If f is an automorphism then the same analysis applies to f^{-1}, so that in particular $|f'(0)| \le 1$ and $|f'^{-1}(0)| \le 1$. But since $f^{-1} \circ f$ is the identity map and $f(0) = 0$, the chain rule gives

$$f'^{-1}(0)f'(0) = 1.$$

Thus $|f'(0)| = 1$ and so f is a rotation $f(z) = e^{i\theta}z$. □

With the Schwarz Lemma in hand, we can find all automorphisms of the disk. Returning to the discussion before the lemma, given an automorphism f of D, let a the element of D such that $f(a) = 0$. Then the composition $f \circ m_{-a}$ is an automorphism of D that fixes 0, and so by the Schwarz Lemma, it is a rotation. Thus (recalling that $m_{-a} = m_a^{-1}$), the general automorphism is

$$f = r_\theta \circ m_a, \qquad r_\theta(z) = e^{i\theta}z, \quad m_a(z) = \frac{z-a}{1-\overline{a}z}.$$

Both r_θ and m_a can be viewed as arising from the group $\mathrm{PSU}_{1,1}(\mathbb{C})$, in the former case because

$$\begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \in \mathrm{SU}_{1,1}(\mathbb{C}).$$

Therefore we have proved that

$$\boxed{\mathrm{Aut}(D) \cong \mathrm{PU}_{1,1}(\mathbb{C}) \cong \mathrm{PSU}_{1,1}(\mathbb{C}).}$$

Again the definition of the general unitary group omits the determinant condition, and so on.

Since the elements $r_\theta \circ m_a$ of $\mathrm{PSU}_{1,1}(\mathbb{C})$ are described by the pairs (θ, a), we have geometrically

$$\mathrm{PSU}_{1,1}(\mathbb{C}) \cong D \times S^1.$$

That is, $\mathrm{PSU}_{1,1}(\mathbb{C})$ can be viewed as a group structure on the open solid torus, a noncompact set.

Let $f : D \longrightarrow D$ be analytic and fix 0. The Schwarz Lemma shows that if f is not an automorphism then $|f'(0)| < 1$. We will quote this fact in proving the Riemann Mapping Theorem.

6. Automorphisms of the Upper Half Plane

Now consider the complex upper half plane,

$$\mathcal{H} = \{x + iy \in \mathbb{C} : y > 0\}.$$

The fractional linear transformation taking 0 to $-i$, i to 0, and ∞ to i is an isomorphism from $\mathcal{H}$ to the disk,

$$t : \mathcal{H} \xrightarrow{\sim} D, \qquad t(z) = \frac{z - i}{-iz + 1}.$$

Therefore the automorphisms of $\mathcal{H}$ and of D are a conjugate groups,

$$\mathrm{Aut}(\mathcal{H}) = t^{-1}\,\mathrm{Aut}(D)\,t.$$

Define the special linear linear group of *real* 2-by-2 matrices,

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\}.$$

Then a matrix calculation shows that for complex $a = a_1 + ia_2$ and $b = b_1 + ib_2$ such that $|a|^2 - |b|^2 = 1$,

$$\frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ b_1 - ib_2 & a_1 - ia_2 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} a_1 + b_2 & b_1 + a_2 \\ b_1 - a_2 & a_1 - b_2 \end{bmatrix}$$

and for real a, b, c, d such that $ad - bc = 1$,

$$\frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a + d + i(b - c) & b + c + i(a - d) \\ b + c - i(a - d) & a + d - i(b - c) \end{bmatrix}$$

so that

$$\frac{1}{2}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\mathrm{SU}_{1,1}(\mathbb{C})\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \mathrm{SL}_2(\mathbb{R}).$$

Therefore,

$$\boxed{\mathrm{Aut}(\mathcal{H}) \cong \mathrm{PGL}_2^+(\mathbb{R}) \cong \mathrm{PSL}_2(\mathbb{R}).}$$

Here $\mathrm{GL}_2^+(\mathbb{R})$ is the group of 2-by-2 real matrices with *positive* determinant. The groups $\mathrm{PGL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{R})$ are not isomorphic because scaling a 2-by-2 real matrix preserves the sign of its determinant.

The group of matrices in $\mathrm{SL}_2(\mathbb{R})$ that fix i is the natural counterpart to the group of matrices in $\mathrm{SU}_{1,1}(\mathbb{C})$ that fix 0, the complex rotation matrices. And indeed this group works out to be the corresponding *real* rotation matrix group, $\mathrm{SO}_2(\mathbb{R})$.

Consider any point of the upper half plane, $z = x + iy$ where $y > 0$. Associate to the point z a real parabolic matrix,

$$p_z = \frac{1}{\sqrt{y}} \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}.$$

Then $p_z(i) = z$. Given any $g \in \mathrm{SL}_2(\mathbb{R})$, let $z = g(i) \in \mathcal{H}$. Then $p_z^{-1}g$ fixes i, and so $g = pk$ where p is real parabolic and $k \in \mathrm{SO}_2(\mathbb{R})$. This establishes the *Iwasawa decomposition* of $\mathrm{SL}_2(\mathbb{R})$,

$$\mathrm{SL}_2(\mathbb{R}) = PK,$$

where

$$P = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ ad = 1 \right\}, \quad K = \mathrm{SO}_2(\mathbb{R}).$$

7. Spaces as Quotient Spaces of Groups

Let G be a locally compact Hausdorff topological group. This means that G is both a group and a topological space and the group operations of multiplication and inversion are continuous under the topology; that every point of G has an open neighborhood with compact closure; that every two distinct points x and y of G lie in disjoint open sets,

Let X be a Hausdorff topological space. Suppose that G acts transitively on X, meaning that G takes any point of X to any other.

Let x be any point of X, and let G_x be the isotropy subgroup of G, meaning the subgroup of G that fixes x. Then there is a natural isomorphism of topological spaces,

$$X \cong G/G_x.$$

Examples:

$$\begin{aligned} S^2 &\cong \mathrm{SO}_3(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \\ \widehat{\mathbb{C}} &\cong \mathrm{SL}_2(\mathbb{C})/P \\ \mathcal{H} &\cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \\ D &\cong \mathrm{SU}_{1,1}(\mathbb{C})/\left\{ \left[\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right] \right\} \end{aligned}$$

The modular curve of level one is

$$Y(1) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}).$$

More generally, for any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the corresponding modular curve is

$$Y = \Gamma \backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}).$$

8. The Hopf Map

(Geometric description of homotopically nontrivial $S^3 \longrightarrow S^2$.)