MATH 311: COMPLEX ANALYSIS — INTEGRATION LECTURE

1. First definition of the complex integral

The data are

- a region $\Omega \subset \mathbf{C}$,
- a continuous function $f : \Omega \longrightarrow \mathbf{C}$,
- a $\mathcal{C}^1$-path $\gamma : [a, b] \longrightarrow \Omega$.

In general, for a complex-valued function over a real interval,

$$\varphi : [a, b] \longrightarrow \mathbf{C}, \qquad \varphi(t) = U(t) + iV(t),$$

the integral of the function over the interval is defined, naturally enough, as

$$\int_{t=a}^{b} \varphi(t)\, dt = \int_{t=a}^{b} U(t)\, dt + i \int_{t=a}^{b} V(t)\, dt,$$

if both integrals exist.

Returning to the data, define

$$\int_\gamma f(z)\, dz = \int_{t=a}^{b} f(\gamma(t))\gamma'(t)\, dt$$

and

$$\int_\gamma f(z)\, |dz| = \int_{t=a}^{b} f(\gamma(t))|\gamma'(t)|\, dt.$$

As a particular case of the second integral, the length of γ is defined as

$$\text{length}(\gamma) = \int_\gamma |dz| = \int_{t=1}^{b} |\gamma'(t)|\, dt.$$

Since the definitions of $\int_\gamma f(z)\, dz$ and $\int_\gamma f(z)\, |dz|$ given just above reduce the complex path-integral to a pair of real integrals, they are convenient for computing. But also, they raise questions.

- Does the integral depend on the parameterization of γ?
- How do the two nonnegative real numbers

$$\left| \int_\gamma f(z)\, dz \right| \qquad \text{and} \qquad \int_\gamma |f(z)|\, |dz|$$

  compare?
- So far, complex path integrals exist only for $\mathcal{C}^1$-paths. Are there any advantages to defining complex path integrals for a larger class of paths?

These questions *can* be answered using only the machinery at hand. To wit:

- A short calculation with the chain rule shows that the integral is invariant under order-preserving reparameterization. That is, all that matters is the direction of path-traversal. As for what happens when the direction is reversed, for any path γ, let $-\gamma$ denote the same path but traversed in the opposite direction. Then unsurprisingly,

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz,$$

but on the other hand,

$$\int_{-\gamma} f(z)\,|dz| = \int_{\gamma} f(z)\,|dz|.$$

- A slightly tricky argument shows that

$$\left|\int_{\gamma} f(z)\,dz\right| \leq \int_{\gamma} |f(z)|\,|dz|.$$

- Defining the complex contour integral for a larger class of paths confers no operational advantage for computing any example that we will care about. However, it does have two advantages. First, often we will deform one path to another, and with integration more broadly defined, we no longer need to worry about whether all the intermediate paths are also $\mathcal{C}^1$. Admittedly, as an argument in favor of generalizing our paths of integration, this may be underwhelming. However, the second advantage is that the answers to the previous two questions become intrinsically clear, rather than seeming to rely on calculations.

2. Second definition of the complex integral

The data are as before, but now γ is assumed only to be continuous. The definitions are

$$\int_{\gamma} f(z)\,dz = \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^{n} f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

and

$$\int_{\gamma} f(z)\,|dz| = \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^{n} f(\gamma(c_j))|\gamma(t_j) - \gamma(t_{j-1})|,$$

if these limits exist. The limit is taken over partitions $P = \{t_j\}$ of $[a, b]$, and corresponding samples $S_P = \{c_j\}$. See the various handouts for details. Here are the key points.

- (*Existence*) If γ is rectifiable and f is continuous then these integrals exist. Proving this requires some fretful detail-mongering, worth the cost only for someone who wants to work on the mathematical techniques of junior-level analysis. By contrast, the integrals in the earlier definition take the form

$$\int_{t=a}^{b} U(t)\,dt + i\int_{t=a}^{b} V(t)\,dt \qquad \text{where } U,\ V \text{ are continuous,}$$

and so they exist by results from real analysis. However, those existence results are essentially as substantive the existence result for the new definitions. So the new definitions aren't really so much more laborious. Somewhere, some time, one must decide whether to think hard about the existence of the integral or to take it for granted.

- (*Invariance*) But granting existence, the integral's invariance under monotonically increasing reparameterizations is essentially wired into the definition. Partitions pass through such reparameterizations, preserving the property of their meshes going to zero. A small technical point here is as follows: Let $r : [a, b] \longrightarrow [c, d]$ be a continuous bijection. Then its inverse is continuous as well. To show this, it suffices to show that r takes closed sets to closed sets. But in this context, closed and compact mean the same thing, and indeed r takes compact sets to compact sets since the continuous image of a compact set is compact.
- (*Size Estimate*) And similarly the estimate $|\int_\gamma f(z)\,dz| \leq \int_\gamma |f(z)|\,|dz|$ is essentially automatic, in consequence of the triangle inequality.
- (*Compatibility*) If γ is again assumed to be $\mathcal{C}^1$ then the two notions of $\int_\gamma f(z)\,dz$ agree, and this is pretty easy to show. Showing that the two notions of $\int_\gamma f(z)\,|dz|$ also agree seems to be harder. (I haven't worked through this carefully yet.) In particular, the two notions of the length of γ agree, but one inequality between them is easier to show than the other. It is a good exercise to think about which inequality should be the easy one, and then to learn its nice geometric proof, which boils down to the Cauchy–Schwarz inequality. By contrast, the proof of the harder inequality is more analytic.