MATH 311: COMPLEX ANALYSIS — TOPOLOGY LECTURE

1. Topology

Engage with Marsden section 1.4 to taste. If things seem unduly complicated, ask me whether they can be simplified (cf. connectedness).

Structures of increasing generality:

$$\begin{aligned}
&(\mathbf{R}, d_{\mathbf{R}}), \quad (\mathbf{C}, d_{\mathbf{C}}) \quad \text{(the line and the plane)}\\
&(\mathbf{R}^n, d_{\mathbf{R}^n}) \quad \text{(Euclidean space)}\\
&(X, d_X) \quad \text{(metric space)}\\
&(X, \mathcal{T}) \quad \text{(topological space)}
\end{aligned}$$

Why generalize?

- Increased scope.
- Sloughing off of spurious detail.

Formulating the apt generalization is much harder than *appreciating* it.

Any subset W of a topological space X itself becomes a topological space in the most natural possible way, with the subspace topology. De Morgan's laws do the little necessary work, e.g.,

$$\bigcup_i \mathcal{O}_{i,W} = \bigcup_i (\mathcal{O}_{i,X} \cap W) = \left(\bigcup_i \mathcal{O}_{i,X} \right) \cap W = \mathcal{O}_X \cap W = \mathcal{O}_W.$$

Definition 1.1 (Topological Continuity)**.** *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $f : X \longrightarrow Y$ be a map. Then f is* **continuous** *if the following condition holds.*

For every open set B in Y, the inverse image $f^{-1}(B)$ is open in X.

The topological definition of continuity subsumes the ε–δ definition in the metric space environment.

Definition 1.2 (Topological Connectedness)**.** *The topological space $(X, \mathcal{T})$ is* **connected** *if the following condition holds.*

X is not the union of two disjoint nonempty subsets.

This definition makes no reference to an ambient space. (By contrast, Marsden's definition is fussy because it takes the ambient space into account.)

Connectedness is a topological property: The continuous image of a connected set is connected. This result subsumes the Intermediate Value Theorem from calculus. The proof is inexorable. With the symbols meaning what they must, if

$$f(X) = \mathcal{O}_1 \cup \mathcal{O}_2$$

then

$$X = f^{-1}(f(X)) = f^{-1}(\mathcal{O}_1 \cup \mathcal{O}_2) = f^{-1}(\mathcal{O}_1) \cup f^{-1}(\mathcal{O}_2).$$

(But this argument raises questions about basic symbol-patterns in connection with set and mappings.)

Definition 1.3 (Topological Compactness)**.** *Let $(X, \mathcal{T})$ be a topological space. Let S be a subset of X. Then S is* **compact** *if the following condition holds.*

Every open cover $\bigcup_i \mathcal{O}_i$ of S has a finite subcover.

Features of this definition in the purely topological setting:

- It is intrinsic, i.e., independent of any ambient space. This means that a topological space is compact with respect to its own topology if and only if it is compact with respect to the topology of any ambient space of which it is a subspace.
- It is a topological property, i.e., it preserved under continuity. In particular, a real-valued function on a compact space assumes extrema. This result subsumes the Extreme Value Theorem from calculus.
- (Tychonoff's Theorem) It is preserved under products. [But what is the product topology?]

The proof that compactness is a topological property is again inexorable. If

$$f(X) = \bigcup_i \mathcal{O}_i$$

then

$$X = f^{-1}(f(X)) = f^{-1}\left(\bigcup_i \mathcal{O}_i\right) = \bigcup_i f^{-1}(\mathcal{O}_i) = \bigcup_{i=1}^n f^{-1}(\mathcal{O}_i),$$

and so

$$f(X) = f\left(\bigcup_{i=1}^n f^{-1}(\mathcal{O}_i)\right) = \bigcup_{i=1}^n f(f^{-1}(\mathcal{O}_i)) = \bigcup_{i=1}^n \mathcal{O}_i.$$

In the metric setting:

- Continuity on compact sets is uniform.

In the Euclidean setting:

- (Heine–Borel Theorem) A subset of Euclidean space is compact if and only if it is closed and bounded.
- (Bolzano–Weierstrass Theorem) A subset of Euclidean space is compact if and only if every sequence in the set has a subsequence that converges in the set.

(Do examples from the Compactness and Uniformity writeup.)