MATH 311: COMPLEX ANALYSIS — TOPOLOGY LECTURE

1. Topology

Engage with Marsden section 1.4 to taste.

Structures of increasing generality:

$$\begin{array}{ll} (\mathbb{R}, d_{\mathbb{R}}),\ (\mathbb{C}, d_{\mathbb{C}}) & \text{(the line and the plane)} \\ (\mathbb{R}^n, d_{\mathbb{R}^n}) & \text{(Euclidean space)} \\ (X, d_X) & \text{(metric space)} \\ (X, \mathcal{T}) & \text{(topological space)} \end{array}$$

Here a *topological space* $(X, \mathcal{T})$ is an ordered pair consisting of a set X and a *topology* $\mathcal{T}$, with the topology a collection of subsets of X, the designated *open* sets. These must have three properties:

(1) The empty set $\emptyset$ and the full space X are open.
(2) Every union $\bigcup_{i \in I} \mathcal{O}_i$ of open sets is again open.
(3) Every finite intersection $\bigcap_{i \in \{i_1, \ldots, i_n\}} \mathcal{O}_i$ of open sets is again open.

The open sets of Euclidean space are the unions of open balls,

$$\mathcal{O} = \bigcup_{i \in I} B(p_i, \varepsilon_i),$$

with each $p_i \in \mathbb{R}^n$, each $\varepsilon_i > 0$, and $B(p_i, \varepsilon_i) = \{x \in \mathbb{R}^n : d(x, p_i) < \varepsilon_i\}$. Similarly for the open sets of any metric space.

Why generalize?

- Increased scope.
- Sloughing off of spurious detail.

Formulating the apt generalization is much harder than *appreciating* it.

Any subset W of a topological space X itself becomes a topological space in the most natural possible way, with the subspace topology. De Morgan's laws do the little necessary work, e.g.,

$$\bigcup_i \mathcal{O}_{i,W} = \bigcup_i (\mathcal{O}_{i,X} \cap W) = \left(\bigcup_i \mathcal{O}_{i,X} \right) \cap W = \mathcal{O}_X \cap W = \mathcal{O}_W.$$

Definition 1.1 (Topological Continuity)**.** *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *be topological spaces, and let* $f : X \longrightarrow Y$ *be a map. Then* f *is* **continuous** *if the following condition holds.*

For every open set B *in* Y*, the inverse image* $f^{-1}(B)$ *is open in* X*.*

The topological definition of continuity subsumes the ε–δ definition in the metric space environment.

Definition 1.2 (Topological Connectedness)**.** *The topological space* $(X, \mathcal{T})$ *is* **connected** *if the following condition holds.*

X *is not the union of two disjoint nonempty open subsets.*

This definition makes no reference to an ambient space. (By contrast, Marsden's definition is busier than this because it takes the ambient space into account.)

Connectedness is a topological property: The continuous image of a connected set is connected. This result subsumes the Intermediate Value Theorem from calculus. The proof is inexorable. With the symbols meaning what they must, if

$$f(X) = \mathcal{O}_1 \sqcup \mathcal{O}_2$$

then

$$X = f^{-1}(f(X)) = f^{-1}\left(\mathcal{O}_1 \sqcup \mathcal{O}_2\right) = f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2).$$

(But this argument raises questions about basic symbol-patterns in connection with set and mappings.)

Definition 1.3 (Topological Compactness). *Let $(X, \mathcal{T})$ be a topological space. Let S be a subset of X. Then S is* **compact** *if the following condition holds.*

Every open cover $\bigcup_i \mathcal{O}_i$ of S has a finite subcover.

Features of this definition in the purely topological setting:

- It is intrinsic, i.e., independent of any ambient space. This means that a topological space is compact with respect to its own topology if and only if it is compact with respect to the topology of any ambient space of which it is a subspace.
- It is a topological property, i.e., it preserved under continuity. In particular, a real-valued function on a compact space assumes extrema. This result subsumes the Extreme Value Theorem from calculus.
- (Tychonoff's Theorem) It is preserved under products. [But what is the product topology?]

The proof that compactness is a topological property is again inexorable. If

$$f(X) = \bigcup_i \mathcal{O}_i$$

then

$$X = f^{-1}(f(X)) = f^{-1}\left(\bigcup_i \mathcal{O}_i\right) = \bigcup_i f^{-1}(\mathcal{O}_i) = \bigcup_{i=1}^n f^{-1}(\mathcal{O}_i),$$

and so

$$f(X) = f\left(\bigcup_{i=1}^n f^{-1}(\mathcal{O}_i)\right) = \bigcup_{i=1}^n f(f^{-1}(\mathcal{O}_i)) = \bigcup_{i=1}^n \mathcal{O}_i.$$

In the metric setting:

- Continuity on compact sets is uniform.

In the Euclidean setting:

- (Heine–Borel Theorem) A subset of Euclidean space is compact if and only if it is closed and bounded.
- (Bolzano–Weierstrass Theorem) A subset of Euclidean space is compact if and only if every sequence in the set has a subsequence that converges in the set.

(Do examples from the Compactness and Uniformity writeup.)