MATH 311: COMPLEX ANALYSIS — COMPLEX NUMBERS LECTURE

1. Complex Numbers

The set of complex numbers is

$$\mathbf{C} = \{x + iy : x, y \in \mathbf{R}\}$$

where

$$i^2 = -1 \quad \text{and} \quad ix = xi \text{ for all } x \in \mathbf{R}.$$

The *real* and *imaginary parts* of a complex number $z = x + iy$ are

$$\begin{aligned} \mathrm{Re}(z) &= x, \\ \mathrm{Im}(z) &= y \qquad \text{(so the imaginary part is real).} \end{aligned}$$

Addition of complex numbers is defined by the formula

$$(x + iy) + (x' + iy') = (x + x') + i(y + y'),$$

and multiplication is defined by the formula

$$(x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y)$$

These make $\mathbf{C}$ an associative ring, though the associativity of multiplication is is a bit messy to verify for now.

The real numbers $\mathbf{R}$ *embed* into the complex numbers $\mathbf{C}$, viz.,

$$\varphi : \mathbf{R} \longrightarrow \mathbf{C}, \qquad \varphi(x) = x + 0i.$$

The embedding is a *ring homomorphism*, meaning it respects the operations of addition and multiplication in the sense that

$$\begin{aligned} \varphi(x + x') &= \varphi(x) + \varphi(x'), \\ \varphi(xx') &= \varphi(x)\varphi(x'). \end{aligned}$$

From now on we replace $\mathbf{R}$ with its embedded image in $\mathbf{C}$.

2. Geometry

Recall that $\mathbf{R}^2$ is a vector space over $\mathbf{R}$, meaning that it comes endowed with addition and scalar multiplication, though it does not come with vector–by–vector multiplication. One can similarly view $\mathbf{C}$ as a vector space over $\mathbf{R}$. The map

$$p : \mathbf{C} \longrightarrow \mathbf{R}^2, \quad x + iy \longmapsto (x, y)$$

respects the vector space operations, meaning that for any complex numbers $z = x + iy$ and $z' = x' + iy'$,

$$\begin{aligned} p(z + z') &= p((x + x') + i(y + y')) \\ &= (x + x', y + y') = (x, y) + (x', y') = p(z) + p(z'), \end{aligned}$$

and that for any real number c and any complex number $z = x + iy$,

$$p(cz) = p(cx + icy) = (cx, cy) = c(x, y) = p(c)p(z).$$

Since p is also a set bijection, it is a linear isomorphism.

But $\mathbf{C}$ also has multiplication, so via the map p we can give $\mathbf{R}^2$ vector–by–vector multiplication as well and think of multiplying complex numbers as multiplying vectors.

Addition in $\mathbf{R}^2$ is defined by the parallelogram law.

For multiplication, use *polar coordinates.* For any nonzero vector (x, y), write

$$(x, y) = \sqrt{x^2 + y^2}\left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right) \overset{\text{call}}{=} r(\cos\theta, \sin\theta)$$

where $r > 0$ is called the *modulus* and $\theta \in [0, 2\pi)$ is called the *argument.* If similarly $(x', y') = r'(\cos\theta', \sin\theta')$ then their product works out to

$$r(\cos\theta, \sin\theta)r'(\cos\theta', \sin\theta') = rr'(\cos(\theta + \theta'), \sin(\theta + \theta')).$$

That is,

the modulus of the product is the product of the moduli

and

the argument of the product is the sum of the arguments.

(But the sum is being taken modulo 2π.)

Return now from $\mathbf{R}^2$ to $\mathbf{C}$, so that the polar decomposition becomes

$$z = r(\cos\theta + i\sin\theta).$$

Thinking of multiplication in polar coordinates makes it easy to extract nth roots. The formula is

$$\big[r(\cos\theta + i\sin\theta)\big]^{1/n} = r^{1/n}\big[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\big],$$
$$k = 0, 1, \dots, n-1.$$

For example, with this formula one can work out that the three cube roots of $8i$ are $-2i$ and $\pm\sqrt{3} + i$. (When possible, one should put answers back into Cartesian form even though the calculations were done in polar form.)

The *exponential notation* for $\cos\theta + i\sin\theta$ is $e^{i\theta}$. In this notation we have shown that

$$re^{i\theta}r'e^{i\theta'} = rr'e^{i(\theta+\theta')}.$$

This *agrees* with the usual rules for exponentiation, but it is not a *consequence* of those rules, since before this they only applied in the context of the real numbers. Also, this formula makes the associativity of multiplication obvious. The formula makes the existence of multiplicative inverses clear as well,

$$(re^{i\theta})^{-1} = r^{-1}e^{-i\theta}$$

since $re^{i\theta}r^{-1}e^{-i\theta} = 1$. The geometric description of the inverse is that the modulus is inverted and the argument is negated. Now it is clear that the complex numbers form a field. Finally, note that $e^{i0} = 1$, $e^{\pm 2\pi i} = 1$, $e^{\pm 4\pi i} = 1$, $\dots$, $e^{2\pi ik} = 1$ for all $k \in \mathbf{Z}$.

3. Complex conjugation and absolute value

The *complex conjugate* map is

$$\overline{} : \mathbf{C} \longrightarrow \mathbf{C}, \qquad \overline{x+iy} = x - iy.$$

This is a ring homomorphism of $\mathbf{C}$ to itself, i.e., $\overline{z+z'} = \overline{z} + \overline{z'}$ and $\overline{zz'} = \overline{z} \cdot \overline{z'}$ for all $z, z' \in \mathbf{C}$. The identities

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}), \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$$

let us suppress the components of complex numbers if we are willing to deal in complex conjugates instead. And of course the identities

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z), \qquad \overline{z} = \mathrm{Re}(z) - i\,\mathrm{Im}(z)$$

show that complex numbers and their conjugates are expressible in terms of components. In particular, the *absolute value* of a complex number can be defined in two equivalent ways,

$$|\quad| : \mathbf{C} \longrightarrow \mathbf{R}_{\geq 0}, \qquad |z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}.$$

This agrees with the old absolute value on $\mathbf{R}$ (recall that we have identified $\mathbf{R}$ with a subset of $\mathbf{C}$) and it agrees with the usual vector absolute value on $\mathbf{R}^2$. It is also the modulus from the previous section.

Some easy identities to verify are

$$\begin{aligned}
|z| &= |\overline{z}|, \\
\pm\mathrm{Re}(z) \leq |\mathrm{Re}(z)| &\leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|, \\
\pm\mathrm{Im}(z) \leq |\mathrm{Im}(z)| &\leq |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|, \\
|re^{i\theta}| &= r, \\
|zz'| &= |z|\,|z'|, \\
|z + z'| &\leq |z| + |z'| \quad \text{(Triangle Inequality)}.
\end{aligned}$$

For the Triangle Inequality: $|z+z'|^2 = (z+z')(\overline{z} + \overline{z'}) = |z|^2 + 2\mathrm{Re}(z\overline{z'}) + |z'|^2 \leq |z|^2 + 2|\mathrm{Re}(z\overline{z'})| + |z'|^2 \leq |z|^2 + 2|z\overline{z'}| + |z'|^2 = (|z| + |z'|)^2$.

The Full Triangle Inequality is

$$\big||z| - |z'|\big| \leq |z \pm z'| \leq |z| + |z'|.$$

These considerations show how write the multiplicative inverse of a nonzero complex number without reference to polar coordinates,

$$z^{-1} = \frac{\overline{z}}{z\overline{z}}.$$

Since the divide is by a positive real we already know it makes sense. In Cartesian coordinates the inverse of $z = x + iy$ is

$$(x+iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

And as before, in polar coordinates the inverse of $z = re^{i\theta}$ is

$$(re^{i\theta})^{-1} = r^{-1}e^{-i\theta}.$$

4. The exponential map

If we *define* for general $z = x + iy$,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

then this is very similar to the polar coordinate map in the sense that

$$|e^{x+iy}| = e^x, \qquad \arg(e^{x+iy}) = y.$$

5. Stereographic Projection

Let $S^2 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere, and let $\mathbf{n}$ denote the north pole $(0, 0, 1)$. Indentify the complex plane $\mathbf{C}$ with the (x, y)–plane in $\mathbf{R}^3$.

Then inside $\mathbf{R}^3$ there is a map, called *stereographic projection*,

$$\pi : S^2 - \mathbf{n} \longrightarrow \mathbf{C}$$

described as follows: place a light source at the north pole $\mathbf{n}$. For any point $(x, y, z) \in S^2 - \mathbf{n}$, consider a light ray emanating downward from $\mathbf{n}$ to pass through the sphere at (x, y, z). The ray continues on to hit the plane, and the point where it hits is designated $\pi(x, y, z)$. The formula works out to

$$\pi(x, y, z) = \frac{x + iy}{1 - z} \qquad \text{for } (x, y, z) \in S^2 - \mathbf{n}.$$

Stereographic projection works out to be conformal.

At times it is handy to think of the sphere instead of the plane. Equivalently, one "adjoins" a point called "∞" to the plane with the understanding that it corresponds to the north pole $\mathbf{n}$ of S^2. Then one speaks about ∞ as as a limit of complex numbers with the tacit understanding that really one is working on the sphere and speaking about $\mathbf{n}$.

6. Imaginary?

One does not have to invoke a square root of -1 to construct the complex numbers from the real numbers. One alternative is the amnesiac approach of forgetting where the multiplication rule for $\mathbf{R}^2$ came from, that is, how we derived the rule

$$(x, y)(x', y') = (xx' - yy', xy' + x'y),$$

and instead pull this out of thin air as a definition of multiplication. One can make the convention that $\mathbf{C}$ simply means $\mathbf{R}^2$ with this multiplication rule. Then purely mechanically,

$$(0, 1)(0, 1) = (-1, 0),$$

giving a square root of -1 in $\mathbf{C}$.

Another approach is to let $\mathbf{C}$ be the ring of polynomials with real coefficients in a formal unknown X, quotiented out by the ideal of polynomials with real coefficients generated by the polynomial $X^2 + 1$. That is, in symbols one can define $\mathbf{C}$ as a quotient ring,

$$\mathbf{R}[X]/\langle X^2 + 1\rangle.$$

The algebraic formalism clears away to the practical rule of manipulating polynomials according to all the usual rules and the additional rule $X^2 + 1 = 0$, again recovering the square root of -1.

7. Field Properties of C

The field **C** has good algebraic and analytic properties.

The natural starting field is the rational numbers **Q**. But thre are two problems:

- **Q** is not *complete*: limits that "ought" to exist in **Q** fail to do so, e.g., $\sqrt{2}$.
- **Q** is not *algebraically closed*: polynomials that "ought" to have solutions in **Q** fail to do so, e.g., $X^2 + 1$.

The smallest complete field containing **Q** is the real numbers **R**. But **R** is not algebraically closed, e.g., $X^2 + 1$ still doesn't have a root in **R**.

The smallest algebraically closed field containing **Q** is denoted $\overline{\mathbf{Q}}$ and simply called the algebraic closure of **Q**. But $\overline{\mathbf{Q}}$ is not complete, e.g., it doesn't contain π. (Note that $\overline{\mathbf{Q}}$ is not a subfield of **R**, e.g., it contains i.

The complex numbers **C** are the smallest algebraically closed complete field over **Q**.