LAURENT SERIES AND SINGULARITIES

1. Introduction

So far we have studied analytic functions. Locally, such functions are represented by power series. Globally, the bounded ones are constant, the ones that get large as their inputs get large are polynomials, and the ones that behave wildly as their inputs get large are transcendental. That is, nonconstant analytic functions diverge tamely or wildly at infinity.

But the Cauchy integral representation formula involves integrands that diverge at a point inside the contour of integration. So next we will study how functions can diverge at a point in the plane. The result will be that they can diverge tamely or wildly at finite points as well. The functions that diverge tamely are called *meromorphic*, and their series expansions are *Laurent series*.

2. Some Handy Formulas

let a and b be complex numbers. If $|a| > |b|$ then

$$\frac{1}{a-b} = \frac{1}{a} \cdot \frac{1}{1-b/a},$$

and so

$$\frac{1}{a-b} = \sum_{n=0}^{\infty} \frac{b^n}{a^{n+1}} = \sum_{n=-1}^{-\infty} \frac{a^n}{b^{n+1}}, \quad |a| > |b|. \tag{1}$$

If $|a| < |b|$ then

$$\frac{1}{a-b} = -\frac{1}{b-a},$$

and so

$$\frac{1}{a-b} = -\sum_{n=0}^{\infty} \frac{a^n}{b^{n+1}} = -\sum_{n=-1}^{-\infty} \frac{b^n}{a^{n+1}}, \quad |a| < |b|. \tag{2}$$

Since

$$\frac{1}{a+b} = \frac{1}{a-(-b)},$$

we also have the formulas

$$\frac{1}{a+b} = \sum_{n=0}^{\infty} (-1)^n \frac{b^n}{a^{n+1}} = -\sum_{n=-1}^{-\infty} (-1)^n \frac{a^n}{b^{n+1}}, \quad |a| > |b|, \tag{3}$$

and

$$\frac{1}{a+b} = \sum_{n=0}^{\infty} (-1)^n \frac{a^n}{b^{n+1}} = -\sum_{n=-1}^{-\infty} (-1)^n \frac{b^n}{a^{n+1}}, \quad |a| < |b|. \tag{4}$$

3. Derivation of Two-sided Series

Consider an annulus in the plane,

$$A = \{z \in \mathbb{C} : R_1 < |z - c| < R_2\}, \qquad 0 \le R_1 < R_2,$$

and consider an analytic function on the annulus,

$$f : A \longrightarrow \mathbb{C}.$$

Even though the centerpoint c doesn't lie in the annulus, f has an expansion in powers of $z - c$. To see this, consider any point $z \in A$. Then z lies between two circles,

$$\gamma_1 = \{\zeta : |\zeta - c| = r_1\} \quad \text{where } R_1 < r_1 < |z - c|,$$

and

$$\gamma_2 = \{\zeta : |\zeta - c| = r_2\} \quad \text{where } |z - c| < r_2 < R_2.$$

Since f is analytic on and between the circles, Cauchy's formula gives

$$f(z) = \frac{1}{2\pi i}\left[\int_{\gamma_2} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \int_{\gamma_1} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}\right] = \frac{1}{2\pi i}\left[I_2 - I_1\right],$$

where

$$I_2 = \int_{\gamma_2} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - c) - (z - c)}$$

and

$$I_1 = \int_{\gamma_1} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - c) - (z - c)}.$$

On γ_2 we have $|\zeta - c| > |z - c|$ and so the handy formula (1) gives

$$I_2 = \sum_{n=0}^{\infty} \int_{\gamma_2} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - c)^{n+1}}\,(z - c)^n.$$

On γ_1 we have $|\zeta - c| < |z - c|$ and so the handy formula (2) gives

$$-I_1 = \sum_{n=-1}^{-\infty} \int_{\gamma_1} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - c)^{n+1}}\,(z - c)^n.$$

(To see clearly that the sum passes through the integral in this case, take its other form from (2), pass that through the integral instead by citing the uniform convergence of geometric series on compacta, and only then change the variable of summation.) Note that in these last two formulas, the integrand is analytic on the entire annulus since the point of singularity has been shifted to the centerpoint c. So the circles of integration can deform to any loop γ in the annulus that winds once counterclockwise about the centerpoint. Thus the two-sided series expansion of f about c is

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n, \qquad a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - c)^{n+1}}}.$$

Note that these extend the formulas arising from power series representation. The only difference is that now n can be a negative integer as well.

Each of the one-sided series $\sum_{n=0}^{\infty} a_n (z - c)^n$ and $\sum_{n=-1}^{-\infty} a_n (z - c)^n$ converges on A because of its description as a contour integral. When the annulus is a punctured disk, the first of these series extends continuously to a_0 at $z = c$, because it is a power series.

The two-sided expansion of f is unique, for if also

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-c)^n,$$

then for any $m \in \mathbb{Z}$,

$$\begin{aligned} 2\pi i b_m &= \sum_{n=-\infty}^{\infty} b_n \int_\gamma \frac{\mathrm{d}\zeta}{(\zeta-c)^{m-n+1}} = \int_\gamma \frac{\sum_{n=-\infty}^{\infty} b_n(\zeta-c)^n \,\mathrm{d}\zeta}{(\zeta-c)^{m+1}} = \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta-c)^{m+1}} \\ &= 2\pi i a_m\,. \end{aligned}$$

4. An Example

Before continuing to develop the general theory, it may be helpful to compute some two-sided series by hand. The handy formulas are more relevant here than the integral formula for the coefficients. Consider the function

$$f(z) = \frac{1}{(z-z_1)(z-z_2)}, \qquad 0 < |z_1| < |z_2|.$$

By partial fractions, this is

$$f(z) = \frac{1}{z_1 - z_2}\left[\frac{1}{z-z_1} - \frac{1}{z-z_2}\right].$$

(In general, the basic partial fractions formula is that for distinct $z_1, \ldots, z_n$,

$$\prod_{j=1}^{n} \frac{1}{z-z_j} = \sum_{j=1}^{n} \frac{a_j}{z-z_j} \quad \text{where} \quad a_j = \prod_{k\neq j} \frac{1}{z_j - z_k}\,.$$

To verify this, it suffices to show that

$$1 = \sum_{j=1}^{n} a_j \prod_{k\neq j}(z-z_k) = \sum_{j=1}^{n} \prod_{k\neq j} \frac{z-z_k}{z_j-z_k}\,.$$

This is a polynomial equation in z of degree at most $n-1$, and it is satisfied by the n distinct values $z_1, \ldots, z_n$, so it holds identically for all z.) Returning to the example, if $|z| < |z_1|$ then formula (2) gives

$$\frac{1}{z-z_1} = -\sum_{n=0}^{\infty} \frac{z^n}{z_1^{n+1}},$$

while if $|z| > |z_1|$ then formula (1) gives

$$\frac{1}{z-z_1} = \sum_{n=-1}^{-\infty} \frac{z^n}{z_1^{n+1}}.$$

Virtually identical calculations apply to the other term $1/(z-z_2)$. Altogether,

$$f(z) = \frac{1}{z_1 - z_2} \cdot \begin{cases} \displaystyle\sum_{n=0}^{\infty} \left(-\frac{1}{z_1^{n+1}} + \frac{1}{z_2^{n+1}} \right) z^n & \text{if } |z| < |z_1|, \\ \displaystyle\sum_{n=-1}^{-\infty} \frac{z^n}{z_1^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{z_2^{n+1}} & \text{if } |z_1| < |z| < |z_2|, \\ \displaystyle\sum_{n=-1}^{-\infty} \left(\frac{1}{z_1^{n+1}} - \frac{1}{z_2^{n+1}} \right) z^n & \text{if } |z_2| < |z|. \end{cases}$$

These are the two-sided series expansions of f in powers of z.

We can also consider powers of $z - z_1$. Here we don't need the partial fractions decomposition of f. The issue is only to expand $1/(z-z_2)$ in powers of $z - z_1$. As usual,

$$\frac{1}{z - z_2} = \frac{1}{(z - z_1) - (z_2 - z_1)}.$$

If $|z - z_1| < |z_2 - z_1|$ then formula (2) gives

$$\frac{1}{z - z_2} = -\sum_{n=0}^{\infty} \frac{(z - z_1)^n}{(z_2 - z_1)^{n+1}} \,,$$

but if $|z - z_1| > |z_2 - z_1|$ then formula (1) gives

$$\frac{1}{z - z_2} = \sum_{n=-1}^{-\infty} \frac{(z - z_1)^n}{(z_2 - z_1)^{n+1}} = \sum_{n=-1}^{-\infty} \frac{(z - z_1)^n}{(z_2 - z_1)^{n+1}} \,,$$

Thus, remembering to multiply by $1/(z - z_1)$,

$$f(z) = \begin{cases} \displaystyle -\sum_{n=-1}^{\infty} \frac{(z - z_1)^n}{(z_2 - z_1)^{n+2}} \,, & \text{if } |z - z_1| < |z_2 - z_1|, \\ \displaystyle \sum_{n=-2}^{-\infty} \frac{(z - z_1)^n}{(z_2 - z_1)^{n+2}} \,, & \text{if } |z - z_1| > |z_2 - z_1|. \end{cases}$$

These are the two-sided series expansions of f in powers of $z - z_1$. The analysis for powers of $z - z_2$ is virtually identical.

5. Classification of Singularities

Now that two-sided series are familiar, we study the consequences of their existence.

Let $f : \Omega \longrightarrow \mathbb{C}$ be analytic. Let $c \notin \Omega$ be a point in the complement of Ω such that some punctured disk $B(c, r) - \{c\}$ lies in Ω. Then f has an *isolated singularity* at c. For example, $f(z) = 1/z$ has an isolated singularity at 0.

As usual, the two-sided series of f at c is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - c)^n, \quad 0 < |z - c| < r.$$

The negatively-indexed terms of the two-sided series make up its *principal part*,

$$pp(z) = \sum_{n=-\infty}^{-1} a_n (z - c)^n.$$

(Note: "principal," not "principle.") There are three possibilities for the principal part of the two-sided series:

- The principal part is zero, i.e., $a_n = 0$ for all $n < 0$. In this case the two-sided series is a power series, and so f extends analytically to $f(c) = a_0$. The singularity of f at c is *removable*.
- The principal part is nonzero but has only finitely many terms. That is, for some positive integer N, $a_n = 0$ for all $n < -N$, and
$$pp(z) = \frac{a_{-N}}{(z-c)^N} + \cdots + \frac{a_{-1}}{z-c}, \quad a_{-N} \neq 0.$$
In this case, the singularity of f at c is a *pole of order* N, and $(z-c)^N f(z)$ has a removable singularity at c, and $(z-c)^N$ is the smallest power of $z-c$ that cancels the pole, making the singularity removable. The function f is *meromorphic* at c. Its series expansion is called a **Laurent** series. By convention, f extends to $f(c) = \infty$. If the original f is viewed as a map to the Riemann sphere rather than the complex plane, then under suitable definitions this extension is continuous and differentiable.
- The principal part has infinitely many nonzero terms. That is, $a_n \neq 0$ for infinitely many $n < 0$. The singularity of f at c is *essential*. For no N does $(z-c)^N f(z)$ extend analytically to c, and there is no sensible way to extend f to a value at c.

For example, the rational function
$$f(z) = \frac{z - z^2}{z}, \quad z \in \mathbb{C} - \{0\}$$
has Laurent expansion $1 - z$ about 0 and hence its singularity at 0 is removable. Defining $f(0) = 1$ extends f to $f(z) = 1 - z$ on all of $\mathbb{C}$. Similarly, the function
$$f(z) = \frac{1}{(z-1)^2} + \frac{5}{z-1} + 12 + 15(z-1)$$
has a pole of order 2 at $c = 1$, and so we define $f(1) = \infty$. But the function
$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \cdots$$
has an essential singularity at 0, and no definition of $f(0)$ makes sense. The series in the previous display is not a Laurent series.

In general, if the function f is analytic on a punctured disk about c and its Laurent series at c is
$$f(z) = \sum_{n=N}^{\infty} a_n (z-c)^n \quad (a_N \neq 0),$$
then N is the *order of vanishing* (or just the *order*) of f at c,
$$\mathrm{ord}_c(f) = N.$$
Note that if f has a pole of order N at c then the order of f at c is not N but $-N$. The order of the zero function is defined as $+\infty$. If g behaves similarly at z then
$$\mathrm{ord}_c(fg) = \mathrm{ord}_c(f) + \mathrm{ord}_c(g).$$
Especially,
$$\mathrm{ord}_c(1/f) = -\mathrm{ord}_c(f).$$

(This relation will do most of problems 5(a) and 5(b) on the homework. For 5(b), just study $c = 0$ and cite periodicity for other $c \in \mathbb{C}$ where f is singular.)

This discussion applies only to *isolated* singularities. A nonisolated singularity is classified as such with no further elaboration.