KLEIN'S j-FUNCTION

Since the j-function can be used to prove Picard's Theorem, we quickly establish some of its properties.

For any lattice $\Lambda \subset \mathbf{C}$ and for any even integer $k \geq 4$, recall the Eisenstein series

$$G_k(\Lambda) = \sum_{\omega \in \Lambda}{}' \frac{1}{\omega^k}.$$

For any such Λ and k, and for any nonzero complex number $m \in \mathbf{C}^\times$, the following homogeneity relation is immediate:

$$G_k(m\Lambda) = m^{-k} G_k(\Lambda).$$

Indeed, lattices are*modules* and homogeneous functions are called *forms*, so that Eisenstein series are among the earliest examples of functions called *modular forms.*

For any $\tau \in \mathcal{H}$, introduce notation for the lattice spanned by τ and 1,

$$\Lambda_\tau = \tau\mathbf{Z} \oplus \mathbf{Z}.$$

Define the Eisenstein series of the variable $\tau \in \mathcal{H}$ to be the corresponding lattice Eisenstein series,

$$G_k(\tau) = G_k(\Lambda_\tau), \quad k \geq 4 \text{ even}.$$

This Eisenstein series of a complex variable satisfies a transformation law. Take any automorphism of $\mathcal{H}$ with integer coefficients,

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Then

$$\begin{aligned} G_k(\tau) &= G_k(\Lambda_\tau) \\ &= G_k(\tau\mathbf{Z} \oplus \mathbf{Z}) \\ &= G_k((a\tau + b)\mathbf{Z} \oplus (c\tau + d)\mathbf{Z}) \\ &= G_k((c\tau + d)\left(\frac{a\tau + b}{c\tau + d}\mathbf{Z} \oplus \mathbf{Z}\right)) \\ &= (c\tau + d)^{-k}\, G_k(\frac{a\tau + b}{c\tau + d}\mathbf{Z} \oplus \mathbf{Z}) \\ &= (c\tau + d)^{-k}\, G_k(\Lambda_{\gamma\tau}) \\ &= (c\tau + d)^{-k}\, G_k(\gamma\tau). \end{aligned}$$

That is, the Eisenstein transformation law is

$$G_k(\gamma\tau) = (c\tau + d)^k\, G_k(\tau), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ \tau \in \mathcal{H}.$$

Next, recall the notational conventions $g_2 = 60G_4$ and $g_3 = 140G_6$. The *discriminant function* is

$$\Delta : \mathcal{H} \longrightarrow \mathbf{C}, \qquad \Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2.$$

The transformation law for the discriminant is

$$\Delta(\gamma\tau) = (c\tau + d)^{12}\,\Delta(\tau), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ \tau \in \mathcal{H}.$$

Klein's j-function is

$$j : \mathcal{H} \longrightarrow \mathbf{C}, \qquad j(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)}.$$

The transformation law for the j-function and the fact that $\mathrm{SL}_2(\mathbf{Z})$ is called the *modular group* combine to explain why j is called the *modular invariant*,

$$\boxed{j(\gamma\tau) = j(\tau), \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ \tau \in \mathcal{H}.}$$

As we have seen before, the change of variable

$$q = e^{2\pi i\tau}$$

maps the upper half plane $\mathcal{H}$ to the unit disk D and has horizontal period one. Also, the condition $\mathrm{Im}(\tau) \to +\infty$ in $\mathcal{H}$ is equivalent to the condition $q \to 0$ in D. It can be shown that

$$j(\tau) \sim \frac{1}{q} \quad \text{as } \mathrm{Im}(\tau) \to +\infty.$$

That is, in some sense j has a simple pole at $i\infty$.

On the other hand, j has no poles in $\mathcal{H}$. To see this, let $\tau \in \mathcal{H}$. Let g_2 and g_3 be the relevant Eisenstein series for the lattice Λ_τ, let $\wp$ be the Weierstrass function for Λ_τ, and consider the cubic polynomial

$$f(x) = 4x^3 - g_2x - g_3 = 4(x - \wp(\tau/2))(x - \wp(1/2))(x - \wp((\tau+1)/2)).$$

The roots $\wp(\tau/2)$, $\wp(1/2)$, and $\wp((\tau+1)/2)$ are distinct because $\wp$ takes these values at least twice each (since $\wp'$ vanishes at half-lattice points), and in general $\wp$ takes all of its values with total multiplicity two. So, the roots of f are distinct, meaning that the discriminant of f is nonzero. But up to a multiplicative constant, the discriminant is

$$\mathrm{disc}(f) = \begin{vmatrix} 4 & 0 & -g_2 & -g_3 & 0 \\ 0 & 4 & 0 & -g_2 & -g_3 \\ 12 & 0 & -g_2 & 0 & 0 \\ 0 & 12 & 0 & -g_2 & 0 \\ 0 & 0 & 12 & 0 & -g_2 \end{vmatrix} = -64(g_2^3 - 27g_3^2) = -64\Delta(\tau).$$

Thus, $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$, and so j has no poles in $\mathcal{H}$.

The set

$$X = \mathrm{SL}_2(\mathbf{Z})\backslash\mathcal{H} \,\cup\, \{i\infty\}$$

can be given the structure of a compact Riemann surface. Once this is done, the j-function is a complex analytic isomorphism from X to the Riemann sphere,

$$j : X \xrightarrow{\sim} \widehat{\mathbf{C}}.$$