INVARIANCE OF THE COMPLEX INTEGRAL

Let Ω be a region in $\mathbf{C}$, let $f : \Omega \longrightarrow \mathbf{C}$ be a continuous function, and consider two rectifiable $\mathcal{C}^0$-curves in Ω,

$$\gamma : [a, b] \longrightarrow \Omega, \qquad \tilde{\gamma} : [a, b] \longrightarrow \Omega.$$

Suppose that $\tilde{\gamma}$ is an orientation-preserving reparameterization of γ, meaning that there exists a continuous increasing bijection

$$r : [a, b] \longrightarrow [c, d]$$

such that

$$\gamma = \tilde{\gamma} \circ r.$$

Let P denote any partition of $[a, b]$ with subordinate sample S,

$$P = \{t_0, \dots, t_n\}, \quad S = \{c_1, \dots, c_n\},$$

and similarly for $[c, d]$,

$$\widetilde{P} = \{\tilde{t}_0, \dots, \tilde{t}_n\}, \quad \widetilde{S} = \{\tilde{c}_1, \dots, \tilde{c}_n\},$$

Then the integrals of f over the two curves are by definition

$$\int_\gamma f(z)\,dz = \lim_{\text{mesh}(P) \to 0} \sum_j f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})).$$

and

$$\int_{\tilde{\gamma}} f(z)\,dz = \lim_{\text{mesh}(\widetilde{P}) \to 0} \sum_j f(\tilde{\gamma}(\tilde{c}_j))(\tilde{\gamma}(\tilde{t}_j) - \tilde{\gamma}(\tilde{t}_{j-1})).$$

This writeup shows that essentially by definition, the two integrals are equal.

The basic idea is that paritition-sample pairs for $[c, d]$ and the partition-sample pairs for $[c, d]$ are in bijective correspondence via r and r^{-1},

$$(\widetilde{P}, \widetilde{S}) = (r(P), r(S)) \quad \text{and} \quad (P, S) = (r^{-1}(\widetilde{P}), r^{-1}(\widetilde{S})).$$

Thus the sets of Riemann sums for the two integrals are the same. For example, each term $\tilde{\gamma}(\tilde{t}_j)$ where $\tilde{t}_j \in \widetilde{P}$ is

$$\tilde{\gamma}(\tilde{t}_j) = \tilde{\gamma}(r(t_j)) = \gamma(t_j) \quad \text{where } t_j \in P,$$

and similarly each $\gamma(t_j)$ where $t_j \in P$ is $\tilde{\gamma}(\tilde{t}_j)$ where $\tilde{t}_j \in \widetilde{P}$.

A small technical point is that the inverse bijection r^{-1} is also continuous. To show this, it suffices to show that r takes closed sets to closed sets. But in this context, closed and compact mean the same thing, and indeed r takes compact sets to compact sets since the continuous image of a compact set is compact.

Now, since r and r^{-1} are uniformly continuous, it follows that if $\{(P_m, S_m)\}$ and $\{(\widetilde{P}_m, \widetilde{S}_m)\}$ are partition-sample sequences related to each other via r and r^{-1} then

$$\lim\{\text{mesh}(P_m) = 0 \iff \lim\{\text{mesh}(\widetilde{P}_m) = 0.$$

These considerations show that the two integrals are equal,

$$\begin{aligned}\int_\gamma f(z)\,dz &= \lim_{\text{mesh}(P)\to 0} \sum_j f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})) \\ &= \lim_{\text{mesh}(\widetilde{P})\to 0} \sum_j f(\tilde\gamma(\tilde c_j))(\tilde\gamma(\tilde t_j) - \tilde\gamma(\tilde t_{j-1})) \\ &= \int_{\tilde\gamma} f(z)\,dz.\end{aligned}$$

If the curves γ and $\tilde\gamma$ are $\mathcal{C}^1$ then the invariance result follows from the chain rule,

$$\begin{aligned}\int_\gamma f(z)\,dz = \int_a^b (f\circ\gamma)\cdot\gamma' &= \int_a^b (f\circ\tilde\gamma\circ r)\cdot(\tilde\gamma\circ r)' \\ &= \int_a^b (f\circ\tilde\gamma\circ r)\cdot(\tilde\gamma'\circ r)\cdot r' \\ &= \int_a^b (((f\circ\tilde\gamma)\cdot\tilde\gamma')\circ r)\cdot r' = \int_c^d (f\circ\tilde\gamma)\cdot\tilde\gamma' = \int_{\tilde\gamma} f(z)\,dz.\end{aligned}$$

But with the argument for $\mathcal{C}^0$-curves in place, even though this calculation is short and natural, it now feels a bit gratuitous.