AN EXISTENCE THEOREM FOR $\int_\gamma f(z)dz$

Let

- Ω be a region in $\mathbf{C}$,
- $f : \Omega \longrightarrow \mathbf{C}$ be a continuous function,
- $\gamma : [a, b] \longrightarrow \Omega$ be a continuous and piecewise $\mathcal{C}^1$ curve.

The *integral of f over γ* was defined in class as

$$\int_\gamma f(z)dz = \int_{t=a}^{b} f(\gamma(t))\gamma'(t)dt,$$

where this integral is broken into finitely many subintegrals as necessary to avoid integrating at the points t such that $\gamma'(t)$ doesn't exist.

While this definition is ideally suited for calculation, a more natural definition of the integral can be made, and it applies in the more general situation where γ is known only to be continuous. Sometimes we may want to deform a path of integration from one piecewise $\mathcal{C}^1$ curve to another. The more general definition eliminates the need to ensure that the intermediate paths are piecewise $\mathcal{C}^1$ as well. More importantly, the general definition makes intrinsically clear some results that the more special definition does not.

We will show that the more natural, general definition of integration is compatible with the definition given above. This requires some terminology:

- A *partition* of the interval $[a, b]$ is a set

  $$P = \{t_0, t_1, \ldots, t_n\}$$

  where

  $$a = t_0 < t_1 < \cdots < t_n = b.$$

  The number $n \geq 0$ can vary from partition to partition. The *mesh of P* is the supremum of the lengths of the subintervals determined by P,

  $$\text{mesh}(P) = \sup_j \{t_j - t_{j-1}\}.$$

  Refining the partition P can not increase its mesh. A *sample subordinate to P* is a set

  $$S_P = \{c_1, \ldots, c_n\}$$

  where

  $$t_0 \leq c_1 \leq t_1 \leq \cdots \leq t_{n-1} \leq c_n \leq t_n.$$

- The *length* of a path $\gamma : [a, b] \longrightarrow \Omega$ is the supremum of the lengths of all of its polygonally inscribed paths,

  $$\text{length}(\gamma) = \sup_P \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})|.$$

  This may be infinite. The curve γ is called *rectifiable* if it has finite length.

Now we can define the integral more generally. Let Ω be a region in $\mathbf{C}$, let $f : \Omega \longrightarrow \mathbf{C}$ and $\gamma : [a,b] \longrightarrow \Omega$ be given, and define the integral of f over γ to be a very general limit, if it exists,

$$\int_\gamma f(z)dz = \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^{n} f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})).$$

The sum on the right is denoted $\Sigma(P, S_P)$.

Existence Theorem. *Let $\Omega \subset \mathbf{C}$ be a region, let $f : \Omega \longrightarrow \mathbf{C}$ be continuous, and let $\gamma : [a,b] \longrightarrow \Omega$ be continuous and rectifiable. Then the integral $\int_\gamma f(z)dz$ exists.*

Proof. Take a sequence of partitions $\{P_1, P_2, \dots, P_N, \dots\}$ such that

$$\lim_N \text{mesh}(P_N) = 0.$$

We need to show that the limit

$$\lim_N \Sigma(P_N, S_{P_N})$$

exists independently of the particular sequence $\{P_N\}$ of partitions used and independently of the sample S_{P_N} chosen for each partition P_N.

Since $[a,b]$ is compact and γ is continuous, the trace of γ,

$$\hat{\gamma} = \{\gamma(t) : t \in [a,b]\}$$

is compact. Consequently f is uniformly continuous on $\hat{\gamma}$. Let $\varepsilon > 0$ be given, and let L denote the length of γ. Then there exists some $\delta > 0$ such that for all $z, z' \in \hat{\gamma}$,

$$|z - z'| < \delta \implies |f(z) - f(z')| < \frac{\varepsilon}{2L}.$$

Also since $[a,b]$ is compact, γ is uniformly continuous on $[a,b]$, so there exists some $\rho > 0$ such that for all $t, t' \in [a,b]$,

$$|t - t'| < \rho \implies |\gamma(t) - \gamma(t')| < \delta.$$

Consequently, for all $t, t' \in [a,b]$,

$$|t - t'| < \rho \implies |f(\gamma(t)) - f(\gamma(t'))| < \frac{\varepsilon}{2L}.$$

Take an index $\widetilde{N}$ large enough that

$$\text{mesh}(P_N) < \rho \quad \text{for any } N > \widetilde{N}.$$

Claim: For any $N > \widetilde{N}$, for any refinement Q of P_N, and for any sample S_Q,

$$|S(P_N, S_{P_N}) - S(Q, S_Q)| < \frac{\varepsilon}{2}.$$

To prove this, note that corresponding to each jth term of $\Sigma(P_N, S_N)$,

$$f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})),$$

we have a jth *sum* of terms in $\Sigma(Q, S_Q)$,

$$\sum_{k=1}^{m} f(\gamma(c'_k))(\gamma(t'_k) - \gamma(t'_{k-1})),$$

with $t'_0 = t_{j-1}$ and $t'_m = t_j$. The jth term $f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1}))$ of $\Sigma(P_N, S_{P_N})$ rewrites as a telescoping sum,

$$\sum_{k=1}^{m} f(\gamma(c_j))(\gamma(t'_k) - \gamma(t'_{k-1})).$$

Thus,

$$\begin{aligned}
&|j\text{th term in } \Sigma(P_N, S_{P_N}) - j\text{th sum in } \Sigma(Q, S_Q)| \\
&\qquad = |\sum_{k=1}^{m}(f(\gamma(c_j)) - f(\gamma(c'_k)))(\gamma(t'_k) - \gamma(t'_{k-1}))| \\
&\qquad \le \sum_{k=1}^{m} |f(\gamma(c_j)) - f(\gamma(c'_k))||\gamma(\tau_k) - \gamma(\tau_{k-1})| \\
&\qquad < \frac{\varepsilon}{2L}\text{length}(\gamma|_{[t_{j-1},t_j]}),
\end{aligned}$$

where we have used the fact that $\text{mesh}(P_N) < \rho$ at the last step. Now sum over j to get

$$|\Sigma(P_N, S_{P_N}) - \Sigma(Q, S_Q)| < \frac{\varepsilon}{2L}\text{length}(\gamma) = \frac{\varepsilon}{2}.$$

This proves the claim.

Now, the claim proves that for any N and M greater than $\tilde{N}$,

$$|\Sigma(P_N, S_{P_N}) - \Sigma(P_M, S_{P_M})| < \varepsilon.$$

To see this, let Q be the common refinement of P_N and P_M, meaning their union, and let S_Q be any corresponding sample. Then by the claim and the triangle inequality,

$$\begin{aligned}
&|\Sigma(P_N, S_{P_N}) - \Sigma(P_M, S_{P_M})| \\
&\quad \le |\Sigma(P_N, S_{P_N}) - \Sigma(Q, S_Q)| + |\Sigma(Q, S_Q) - \Sigma(P_M, S_{P_M})| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This pretty much proves the theorem in turn. We now know that the complex sequence

$$\{\Sigma(P_N, S_{P_N})\}$$

is a Cauchy sequence. By the completeness of $\mathbf{C}$, the sequence converges to some complex number

$$\lim_N \{\Sigma(P_N, S_{P_N})\}.$$

What remains to be shown is that this limit is independent of which sequence of partitions P_N we chose and of which subordinate sample S_{P_N} we chose for each P_N. But if $\{P'_N\}$ is another such sequence of partitions, or the same sequence but with different samples, then the complex sequence

$$\{\Sigma(P'_N, S_{P'_N})\}$$

also converges, to some complex number

$$\lim_N \{\Sigma(P'_N, S_{P'_N})\}.$$

The blended sequence

$$\{\Sigma(P_1, S_{P_1}), \Sigma(P'_1, S_{P'_1}), \Sigma(P_2, S_{P_2}), \Sigma(P'_2, S_{P'_2}), \dots\}$$

again converges since the meshes of the blended sequence of partitions go to 0, and its limit must be both of the previous limits because each is the limit of a subsequence. Thus the limit has the desired independence properties, making it a suitable definition of the integral. □