THE BASIC COMPLEX INTEGRAL ESTIMATE

Let Ω be a region, let $f : \Omega \longrightarrow \mathbf{C}$ be a continuous function, and let $\gamma : [a, b] \longrightarrow \Omega$ be a rectifiable path. Then

$$\left| \int_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\, |dz|.$$

To see this, recall that the integral is the limit of Riemann sums, and compute (using the fact that the absolute value function is continuous and using the triangle inequality) that

$$\begin{aligned}
\left| \int_\gamma f(z)\, dz \right| &= \left| \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^{n} f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})) \right| \\
&= \lim_{\text{mesh}(P)\to 0} \left| \sum_{j=1}^{n} f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})) \right| \\
&\le \lim_{\text{mesh}(P)\to 0} \sum_{j=1}^{n} |f(\gamma(c_j))|\, |(\gamma(t_j) - \gamma(t_{j-1}))| \\
&= \int_\gamma |f(z)|\, |dz|.
\end{aligned}$$

Truly, this is all there is to it.

For the skeptics, here also is the standard argument (in our text, for example) given the stronger hypothesis that γ is a $\mathcal{C}^1$-path. This argument is required if one has defined the complex integral *only* for (piecewise) $\mathcal{C}^1$-paths, by parameterization, rather than for rectifiable curves, by Riemann sums. Note that the following proof, despite proceeding from stronger hypotheses than the previous one, is more complicated. In this instance, avoiding the Riemann sum definition of the integral has made things harder rather than easier.

The first idea is that the comparable result in the *real* setting,

$$\left| \int_{t=a}^{b} g(t)\, dt \right| \le \int_{t=a}^{b} |g(t)|\, dt,$$

is easy: simply integrate the relation $-|g(t)| \le g(t) \le |g(t)|$ over the interval $[a, b]$. So the complex case seems as though it should be easy too. But reducing the complex case to the real case poses two obstacles. First, $f(z)$ takes complex values, and second, dz is a complex differential. The following argument addresses these issues one at a time, first reducing the problem to the case of a complex-valued function to that of a real-valued one, assuming that the differential is real-valued, and then using the definitions $dz = \gamma'(t)\, dt$, $|dz| = |\gamma'(t)|\, dt$ to reduce the case of a complex differential to that of a real one.

So, first consider a continuous complex-valued function on a real interval,

$$\varphi : [a, b] \longrightarrow \mathbf{C}.$$

The claim is that

$$\left| \int_{t=a}^{b} \varphi(t)\, dt \right| \le \int_{t=a}^{b} |\varphi(t)|\, dt.$$

To see this, let $\int_{t=a}^{b} \varphi(t)\, dt = re^{i\theta}$. (If the integral is zero then there is nothing to prove, so we may assume that the integral is nonzero, making the polar decomposition well defined.) Then

$$\begin{aligned} \left| \int_{t=a}^{b} \varphi(t)\, dt \right| &= r = e^{-i\theta} re^{i\theta} = e^{-i\theta} \int_{t=a}^{b} \varphi(t)\, dt = \int_{t=a}^{b} e^{-i\theta} \varphi(t)\, dt \\ &= \int_{t=a}^{b} (\mathrm{Re}(e^{-i\theta}\varphi(t)) + i\mathrm{Im}(e^{-i\theta}\varphi(t)))\, dt. \end{aligned}$$

But the integral is real, so its imaginary part must be zero, leaving us in a position to quote the inequality from the real case and then quote the fact that the size of the real component is at most the size of the complex number,

$$\begin{aligned} \left| \int_{t=a}^{b} \varphi(t)\, dt \right| = \int_{t=a}^{b} \mathrm{Re}(e^{-i\theta}\varphi(t))\, dt &\le \int_{t=a}^{b} |\mathrm{Re}(e^{-i\theta}\varphi(t))|\, dt \\ &\le \int_{t=a}^{b} |e^{-i\theta}\varphi(t)|\, dt = \int_{t=a}^{b} |\varphi(t)|\, dt. \end{aligned}$$

The result follows. Let $\varphi = (f \circ \gamma) \cdot \gamma'$. Then

$$\begin{aligned} \left| \int_{\gamma} f(z)\, dz \right| = \left| \int_{t=a}^{b} \varphi(t)\, dt \right| &\le \int_{t=a}^{b} |\varphi(t)|\, dt \\ &= \int_{t=a}^{b} |f(\gamma(t))|\, |\gamma'(t)|\, dt = \int_{\gamma} |f(z)|\, |dz|. \end{aligned}$$