A COMPATIBILITY VERIFICATION FOR THE COMPLEX INTEGRAL

Let Ω be a region in $\mathbf{R}^2$, and let $f : \Omega \longrightarrow \mathbf{C}$ be a continuous function, and let $\gamma : [a, b] \longrightarrow \Omega$ be a $\mathcal{C}^1$ path. Then we have two definitions of the integral of f over γ, the first using the derivative of γ,

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \lim_{\mathrm{mesh}(P)\to 0} \sum_j f(\gamma(c_j))\gamma'(c_j)(t_j - t_{j-1})\end{aligned}$$

and the second making no reference to the derivative,

$$\int_\gamma f(z)\,dz = \lim_{\mathrm{mesh}(P)\to 0} \sum_j f(\gamma(c_j))(\gamma(t_j) - \gamma(t_{j-1})).$$

This handout sketches the argument that the definitions are compatible.

Part of the summand in the first definition is

$$\gamma'(c_j)(t_j - t_{j-1}) = \left(u'(c_j) + iv'(c_j)\right)(t_j - t_{j-1}),$$

while two applications of the Mean Value Theorem show that part of the summand in the second definition is

$$\gamma(t_j) - \gamma(t_{j-1}) = \left(u'(d_j) + iv'(e_j)\right)(t_j - t_{j-1}), \quad \text{for some } d_j, e_j \in [t_{j-1}, t_j].$$

Thus the difference of the summands is

$$f(\gamma(c_j))\left(u'(c_j) - u'(d_j) + i(v'(c_j) - v'(e_j))\right)(t_j - t_{j-1}).$$

Since f is continuous and the trace of γ is compact (it is the continuous image of the compact set $[a, b]$), f is bounded on the trace of γ. Also, since γ is $\mathcal{C}^1$, the component function derivatives u' and v' are continuous on $[a, b]$, and since $[a, b]$ is compact, they are uniformly continuous there. Therefore, given $\varepsilon > 0$, if the partition P is fine enough then for each j,

$$|f(\gamma(c_j))\left(u'(c_j) - u'(d_j) + i(v'(c_j) - v'(e_j))\right)(t_j - t_{j-1})| < \frac{\varepsilon(t_j - t_{j-1})}{b - a}.$$

This makes the two sums within ε of each other. Thus the integrals are equal.