A COMPATIBILITY VERIFICATION FOR PATH-LENGTH

Let Ω be a region in $\mathbf{R}^2$, and let $\gamma : [a,b] \longrightarrow \Omega$ be a $\mathcal{C}^1$ path. Then we have two definitions of the length of γ, the first using the derivative of γ,

$$\operatorname{length}(\gamma) = \int_a^b |\gamma'(t)|\,dt,$$

and the second being the supremum of inscribed polygonal path-lengths, making no reference to the derivative,

$$\operatorname{length}(\gamma) = \sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})|.$$

This handout shows that the definitions are compatible.

It is not hard to establish that the integral of $|\gamma'|$ is at least the length of any inscribed polygonal path, since these lengths grow under refinement and the integral is conceptually their limit. Indeed, take a partition of $[a,b]$,

$$P = \{t_0, t_1, \dots, t_n\},$$

and assume that no consecutive pair of division points t_{j-1} and t_j have the same image under γ. (If $\gamma(t_{j-1}) = \gamma(t_j)$ then the pair contributes nothing to the length of the polygonal path, and so we may drop its second point from the overall calculation.) Fix any $j \in \{1, \dots, n\}$. Consider the unit vector in the direction between the jth pair of consecutive polygon points,

$$\hat{v} = (\gamma(t_j) - \gamma(t_{j-1}))/|\gamma(t_j) - \gamma(t_{j-1})|.$$

The trivial estimate $\hat{v} \cdot \gamma'(t) \le |\hat{v} \cdot \gamma'(t)|$, then the Cauchy–Schwarz inequality $|\hat{v} \cdot \gamma'(t)| \le |\hat{v}|\,|\gamma'(t)| = |\gamma'(t)|$ give the inequality in the calculation

$$\begin{aligned} |\gamma(t_j) - \gamma(t_{j-1})| &= \hat{v} \cdot (\gamma(t_j) - \gamma(t_{j-1})) \\ &= \hat{v} \cdot \int_{t_{j-1}}^{t_j} \gamma'(t)\,dt = \int_{t_{j-1}}^{t_j} \hat{v} \cdot \gamma'(t)\,dt \le \int_{t_{j-1}}^{t_j} |\gamma'(t)|\,dt. \end{aligned}$$

That is, the jth inscribed polygonal segment length is at most the jth piece of the integral. Sum over j to get the inequality

$$\sum_j |\gamma(t_j) - \gamma(t_{j-1})| \le \int_a^b |\gamma'(t)|\,dt.$$

This holds for any partition P, and the right side is independent of P. It follows that

$$\sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})| \le \int_a^b |\gamma'(t)|\,dt. \tag{1}$$

The opposite inequality is more delicate. The idea is to get polygonal path-lengths as close to the integral of $|\gamma'|$ as desired. The argument to follow can prove both inequalities, but the easier direction deserved its correspondingly smoother proof.

The derivatives $x'(t)$ and $y'(t)$ of the component functions of $\gamma(t)$ are continuous, and their domain $[a,b]$ is compact, so they are uniformly continuous on their domain. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ so that

$$\left\{\begin{matrix} t, \tilde{t} \in [a,b], \\ |\tilde{t} - t| < \delta \end{matrix}\right\} \implies \sup\{|x'(\tilde{t}) - x'(t)|, |y'(\tilde{t}) - y'(t)|\} < \frac{\varepsilon}{4(b-a)}.$$

So if P partitions $[a,b]$ more finely than δ then for any $j \in \{1, \dots, n\}$ and for any $s_j, \tilde{s}_j \in [t_{j-1}, t_j]$, the reverse triangle inequality gives

$$\begin{aligned} \big|\big(x'(s_j), y'(\tilde{s}_j)\big)\big| &= \big|\big(x'(t_j), y'(t_j)\big) \;-\; \big(x'(t_j) - x'(s_j), y'(t_j) - y'(\tilde{s}_j)\big)\big| \\ &\geq \big|\big(x'(t_j), y'(t_j)\big)\big| \;-\; \big|\big(x'(t_j) - x'(s_j), y'(t_j) - y'(\tilde{s}_j)\big)\big| \\ &> \big|\big(x'(t_j), y'(t_j)\big)\big| - \frac{\varepsilon}{2(b-a)} \\ &= |\gamma'(t_j)| - \frac{\varepsilon}{2(b-a)}. \end{aligned}$$

Now compute for any j, using the Mean Value Theorem twice at the first step and using the previous calculation at the last step, that

$$\begin{aligned} |\gamma(t_j) - \gamma(t_{j-1})| &= \big|\big(x'(s_j), y'(\tilde{s}_j)\big)\,(t_j - t_{j-1})\big| \quad \text{for some } s_j, \tilde{s}_j \in [t_{j-1}, t_j] \\ &= \big|\big(x'(s_j), y'(\tilde{s}_j)\big)\big|\,(t_j - t_{j-1}) \\ &> |\gamma'(t_j)|\,(t_j - t_{j-1}) - \frac{\varepsilon}{2(b-a)}(t_j - t_{j-1}). \end{aligned}$$

Sum over j to get

$$\sum_j |\gamma(t_j) - \gamma(t_{j-1})| > \sum_j |\gamma'(t_j)|(t_j - t_{j-1}) - \frac{\varepsilon}{2}.$$

But if the partition is fine enough, then by the definition of the Riemann integral we also have

$$\sum_j |\gamma'(t_j)|(t_j - t_{j-1}) > \int_a^b |\gamma'(t)|\,dt - \frac{\varepsilon}{2}.$$

Combining the last two displays gives

$$\sum_j |\gamma(t_j) - \gamma(t_{j-1})| > \int_a^b |\gamma'(t)|\,dt - \varepsilon,$$

and it follows trivially that

$$\sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})| > \int_a^b |\gamma'(t)|\,dt - \varepsilon.$$

Since this holds for all $\varepsilon > 0$,

$$\sup_P \sum_j |\gamma(t_j) - \gamma(t_{j-1})| \geq \int_a^b |\gamma'(t)|\,dt. \tag{2}$$

Equations (1) and (2) together give the result.