MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 9

Problems:

1. Let D be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$, let ∂D be its boundary circle, and let G be the group of all fractional linear transformations mapping D to itself. Thus

$$G = \{R_\theta \circ T_a : \theta \in \mathbb{R}/2\pi\mathbb{Z}, a \in D\},$$

where $R_\theta(z) = e^{i\theta}z$ and $T_a(z) = \dfrac{z-a}{1-\overline{a}z}$. Note that T_a maps $a \mapsto 0$ and $0 \mapsto -a$, and that $T_a^{-1} = T_{-a}$.

(a) Show that for any $a, b \in D$, there is an element $T \in G$ such that $Ta = b$.

(b) Show that for any points $a_1, a_2, b_1, b_2 \in D$, there is an element $T \in G$ such that $Ta_1 = b_1$ and $Ta_2 = b_2$ if and only if

$$\left|\frac{a_1 - a_2}{1 - a_1\overline{a}_2}\right| = \left|\frac{b_1 - b_2}{1 - b_1\overline{b}_2}\right|,$$

and that if there is such a T then it is unique. (Suggestion: A map $T \in G$ takes $a_1 \mapsto b_1$ and $a_2 \mapsto b_2$ exactly when $T_{b_2} \circ T \circ T_{a_2}^{-1}$ takes $0 \mapsto 0$ and $T_{a_2}(a_1) \mapsto T_{b_2}(b_1)$.)

(c) Show that for any points $a, b \in D$ there is a unique circle C passing through a and b and meeting ∂D orthogonally. (Suggestion: Prove first that the only circles through 0 and perpendicular to ∂D are diameters of D.)

2. Let D and G be as above. D will be called the non-Euclidean plane and G will be called the group of non-Euclidean motions. Any circular arc in D that meets ∂D orthogonally will be called a non-Euclidean line. For brevity, non-Euclidean will be replaced by nE.

(a) Show that any two distinct points in D determine a unique nE line. Show that any two nE lines either meet in a single point or in no points; in the latter case they will be called parallel. Show that nE motions preserve the angles of intersections of nE lines.

(b) Find a point $a \in D$ and a nE line L not passing through a such that there is more than one line through a parallel to L. (The axioms of Euclidean geometry other than the parallel postulate are satisfied by nE geometry.)

(c) Show that the sum of the angles of a nE triangle is at most π. (Suggestion: Use a nE motion to put the nE triangle in a convenient position.) Can the sum of the angles ever equal π? How small can the sum of the angles actually be?

(d) Show that the function $d(a, b) = \left|\dfrac{a-b}{1-\overline{a}b}\right|$ behaves somewhat like a nE distance in the sense that $d(a_1, a_2) = d(b_1, b_2)$ if and only if there is a nE motion taking a_1 to b_1 and a_2 to b_2. Describe nE circles. (A true distance should have the additional properties that $d(a, b) + d(b, c) = d(a, c)$ when a, b, c are nE collinear, and that $d(a, b) + d(b, c) \geq d(a, c)$ for all points. In case you are curious, $\delta(a, b) = 1/2 \tanh^{-1}(d(a, b))$ is a true distance.)

3. There is a fractional linear transformation taking the unit disk to the upper half plane (what is it?), and so there is another model of nE geometry based on the upper half plane. What are the nE straight lines in this model?