

MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 8

Reading: Marsden, sections 5.1, 5.2.

Problems:

1. Consider a nonidentity fractional linear transformation

$$Sz = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (z), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}_2(\mathbf{C}).$$

Show that S can be classified as elliptic, parabolic, hyperbolic, or loxodromic solely in terms of the projective trace $\pm(a+d)$, which is well defined. For example, S is elliptic if and only if $a+d$ is real and $|a+d| < 2$.

2. The *cross-ratio* of any four distinct points $z_1, z_2, z_3, z_4 \in \mathbf{C}$ is defined as

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}.$$

(a) Show that $(z_1, z_2, z_3, z_4) = Sz_1$, where S is the unique fractional linear transformation such that $Sz_2 = 1, Sz_3 = 0, Sz_4 = \infty$.

(b) Show that $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any fractional linear transformation T .

(c) Show that (z_1, z_2, z_3, z_4) is real precisely when these four points lie either on the same line or the same circle.

3. (a) Show that the reflection z^* of the point z in the circle passing through three distinct points z_1, z_2, z_3 is characterized by the condition

$$(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}.$$

(Hint for one direction: If the circle through z_1, z_2, z_3 is centered at a and has radius R then using properties of the cross-ratio shows that

$$\begin{aligned} \overline{(z, z_1, z_2, z_3)} &= \overline{(z - a, z_1 - a, z_2 - a, z_3 - a)} \\ &= (\bar{z} - \bar{a}, R^2/(z_1 - a), R^2/(z_2 - a), R^2/(z_3 - a)) \\ &= (R^2/(\bar{z} - \bar{a}), z_1 - a, z_2 - a, z_3 - a) \\ &= (R^2/(\bar{z} - \bar{a}) + a, z_1, z_2, z_3). \end{aligned}$$

Since I am giving you the calculation, explain the steps.)

(b) Suppose that z^* is the reflection of z in a circle C . Show that Sz^* is the reflection of Sz in the circle SC .

4. Find fractional linear transformations:

(a) R , taking the half-disk $\{z \in \mathbf{C} : |z| < 1, \text{Im}(z) > 0\}$ to the first quadrant $\{z \in \mathbf{C} : \text{Re}(z) > 0, \text{Im}(z) > 0\}$.

(b) S , taking the circle $|z| = 2$ to the circle $|z + 1| = 1$, the point -2 to the origin, and the origin to i .

(c) T , taking the circles $|z| = 1$ and $|z - 1/4| = 1/4$ to $|z| = R$ and $|z| = 1$ respectively, with $R > 1$, and taking $1/2 \mapsto 1$. What must R be?