MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 3

Reading: Marsden, sections 2.1, 2.2, 2.3.

Problems:

1. Evaluate \( \int_{-i}^{i} |z| \, dz \) by taking the path (a) rectilinearly, (b) along the left half of the unit circle, (c) along the right half of the unit circle.

2. Compute \( \int_{|z|=1} \overline{z} \, dz \). (By convention, when a simple closed path is specified as a point set, traverse it once counterclockwise.)

3. Compute \( \int_{\gamma} \frac{dz}{z^2 - 1} \), where \( \gamma \) is the closed curve comprising the right half of \( \{|z| = 2\} \) and \( \{0 + iy : -2 \leq y \leq 2\} \), traversed once counterclockwise. (Hint: partial fractions and the Deformation Theorem.)

4. Suppose that \( \gamma \) is a simple closed curve, and that \( f(z) \) is an analytic function in a region containing \( \gamma \); suppose moreover that the derivative \( f'(z) \) is continuous in that region. Show that \( \int_{\gamma} f(z) f'(z) \, dz \) is purely imaginary.

5. Recall from advanced calculus Green’s theorem: if \( \gamma \) is the smooth oriented boundary of an open subset \( \Omega \) of the plane and if \( u, v \) are continuously differentiable real-valued functions in an open neighborhood of the closure of \( \Omega \) then

\[
\int_{\gamma} u \, dx + v \, dy = \int_{\Omega} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx \, dy.
\]

(a) Show that Green’s theorem can be restated: if \( \gamma, \Omega \) are as above and \( f \) is a complex-valued continuously differentiable (with respect to \( x \) and \( y \)) function in an open neighborhood of the closure of \( \Omega \) then

\[
\int_{\gamma} f(z) \, dz = 2i \int_{\Omega} \frac{\partial f}{\partial \bar{z}} \, dx \, dy,
\]

where as usual, \( \partial / \partial \bar{z} = 1/2(\partial / \partial x + i \partial / \partial y) \).

(b) Use Green’s theorem as restated in part (a) to prove Cauchy’s theorem under the hypothesis that \( f(z) \) is analytic AND has continuous partial derivatives with respect to \( x \) and \( y \). Is this result stronger or weaker than the complex-analytic statement of Cauchy’s theorem?