Reading: Marsden, sections 2.1, 2.2, 2.3.

Problems:

1. Evaluate $\int_{-i}^{i} |z| \, dz$ by taking the path (a) rectilinearly, (b) along the left half of the unit circle, (c) along the right half of the unit circle.

2. Compute $\int_{|z|=1} z \, dz$. (By convention, when a simple closed path is specified as a point set, traverse it once counterclockwise.)

3. Compute $\int_{\gamma} dz/(z^2 - 1)$, where $\gamma$ is the closed curve comprised of the right half of $\{|z| = 2\}$ and $\{0 + iy : -2 \leq y \leq 2\}$, traversed once counterclockwise. (Hint: partial fractions and the Deformation Theorem.)

4. Suppose that $\gamma$ is a simple closed curve, and that $f(z)$ is an analytic function in a region containing $\gamma$; suppose moreover that the derivative $f'(z)$ is continuous in that region. Show that $\int_{\gamma} f(z) f'(z) \, dz$ is purely imaginary.

5. Recall from advanced calculus Green’s theorem: if $\gamma$ is the smooth oriented boundary of an open subset $\Omega$ of the plane and if $u, v$ are continuously differentiable real-valued functions in an open neighborhood of the closure of $\Omega$ then

$$\int_{\gamma} u \, dx + v \, dy = \int_{\Omega} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx \, dy.$$ 

(a) Show that Green’s theorem can be restated: if $\gamma, \Omega$ are as above and $f$ is a complex-valued continuously differentiable (with respect to $x$ and $y$) function in an open neighborhood of the closure of $\Omega$ then

$$\int_{\gamma} f(z) \, dz = 2i \int_{\Omega} \frac{\partial f}{\partial \bar{z}} \, dx \, dy,$$

where as usual, $\partial / \partial \bar{z} = 1/2(\partial / \partial x + i \partial / \partial y)$.

(b) Use Green’s theorem to prove Cauchy’s integral theorem under the hypothesis that $f(z)$ is analytic AND has continuous partial derivatives with respect to $x$ and $y$. Is this result stronger or weaker than the complex-analytic statement of Cauchy’s integral theorem?