MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 3

Reading: Marsden, sections 2.1, 2.2, 2.3.

Problems:

1. Evaluate $\int_{-i}^{i} |z|\,\mathrm{d}z$ by taking the path (a) rectilinearly, (b) along the left half of the unit circle, (c) along the right half of the unit circle.

2. Compute $\int_{|z|=1} \overline{z}\,\mathrm{d}z$. (By convention, when a simple closed path is specified as a point set, traverse it once counterclockwise.)

3. Compute $\int_\gamma \mathrm{d}z/(z^2-1)$, where γ is the closed curve comprising the right half of $\{|z|=2\}$ and $\{0+iy : -2 \le y \le 2\}$, traversed once counterclockwise. (Hint: partial fractions and the Deformation Theorem.)

4. Suppose that γ is a simple closed curve, and that $f(z)$ is an analytic function in a region containing γ; suppose moreover that the derivative $f'(z)$ is continuous in that region. Show that $\int_\gamma \overline{f(z)} f'(z)\,\mathrm{d}z$ is purely imaginary.

5. Recall from advanced calculus Green's theorem: if γ is the smooth oriented boundary of an open subset Ω of the plane and if u, v are continuously differentiable real-valued functions in an open neighborhood of the closure of Ω then

$$\int_\gamma u\,\mathrm{d}x + v\,\mathrm{d}y = \int_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

(a) Show that Green's theorem can be restated: if γ, Ω are as above and f is a complex-valued continuously differentiable (with respect to x and y) function in an open neighborhood of the closure of Ω then

$$\int_\gamma f(z)\,\mathrm{d}z = 2i\int_\Omega \frac{\partial f}{\partial \overline{z}}\,\mathrm{d}x\,\mathrm{d}y,$$

where as usual, $\partial/\partial\overline{z} = 1/2(\partial/\partial x + i\,\partial/\partial y)$.

(b) Use Green's theorem as restated in part (a) to prove Cauchy's theorem under the hypothesis that $f(z)$ is analytic AND has continuous partial derivatives with respect to x and y. Is this result stronger or weaker than the complex-analytic statement of Cauchy's theorem?