Reading: Marsden, sections 1.3, 1.5, 1.6.

Problems:

1. Show that if \( f(z) \) is analytic in a region \( \Omega \) and either \( \text{Re} f(z) \) or \( |f(z)| \) is constant in \( \Omega \) then \( f(z) \) must be constant there.

2. Show that if \( f(z) \) is analytic and its second partial derivatives exist and are continuous then \( \Delta (|f(z)|^2) = 4|f'(z)|^2 \), where \( \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 \).

3. Show that the function

\[
f(z) = \begin{cases} 
  e^{-1/z^4} & \text{for } z \neq 0 \\
  0 & \text{for } z = 0 
\end{cases}
\]

is analytic at all \( z \neq 0 \), is not analytic at \( z = 0 \), but satisfies the Cauchy–Riemann equations at \( z = 0 \). (For \( z \neq 0 \), decompose \( f \) as a composition of functions each known to be analytic; to show \( f \) is not analytic at 0 it suffices to show \( f \) is not even continuous at 0; the form \( f_x = -if_y \) of the Cauchy–Riemann equations is easiest to check at 0.)

4. Extend the trigonometric functions to complex arguments by defining

\[
\sin z = \sin x \cosh y + i \cos x \sinh y \\
\cos z = \cos x \cosh y - i \sin x \sinh y
\]

for all \( z = x + iy \in \mathbb{C} \); here the hyperbolic functions are defined as usual by

\[
\sinh y = \frac{e^y - e^{-y}}{2}, \cos y = \frac{e^y + e^{-y}}{2} \text{ for } y \in \mathbb{R}.
\]

(a) Show that \( \sin z \) is analytic for all \( z \in \mathbb{C} \) and find all points \( z \) for which \( \sin z = 0 \).

(b) Show:

\[
\cos(\pi/2 - z) = \sin z, \cos(z + \pi) = -\cos z, \cos(z + 2\pi) = \cos z, \\
\sin^2 z + \cos^2 z = 1, \cos z = (e^{iz} + e^{-iz})/2 \text{ for all } z \in \mathbb{C}.
\]

(c) Discuss the mapping described by the function \( w = \cos z \). It suffices to consider the strip \( 0 \leq \text{Re} z \leq 2\pi \), by periodicity. The cosine function can be written as the composition of the functions \( z \to iz, z \to e^z, z \to (z + z^{-1})/2 \), each of which is familiar. Illustrate suitable restrictions of each of these functions to give a good sense of the composite. On what part of the strip is \( \cos \) 1-to-1? What is its output?

(d) Discuss a single-valued inverse cosine function: where it is defined, a formula for it in terms of the logarithm function, what its branch points are.