

## HEISENBERG UNCERTAINTY

Consider a wavefunction

$$f : \mathbf{R} \longrightarrow \mathbf{C}.$$

Assume that  $f$  is well-behaved, so for example  $f$  might be a Schwartz function. For future convenience, introduce the notation  $\langle \cdot, \cdot \rangle$  for the inner product, i.e.,

$$\langle f, g \rangle \stackrel{\text{def}}{=} \int f(x)\overline{g(x)} dx \quad \text{for suitable functions } f \text{ and } g$$

(all integrals here are taken over  $\mathbf{R}$ ), and similarly for the  $L^2$ -norm,

$$\|f\|_2 \stackrel{\text{def}}{=} \sqrt{\langle f, f \rangle} = \sqrt{\int |f(x)|^2 dx} \quad \text{for suitable functions } f.$$

In particular, the  $L^2$ -norm of  $f$  is 1,

$$\|f\|_2 = 1.$$

Consider the operators

$$A = i \frac{d}{dx}, \quad B = x \text{ (multiplication by } x).$$

Their Lie bracket is

$$(AB - BA)f = i \frac{d}{dx}(xf) - ix \frac{d}{dx}f = if + ix f' - ix f' = if.$$

Or, more concisely, as a relation of operators,

$$-i[A, B] = \text{id}.$$

And in fact, all that we need is the weaker relation that  $A$  and  $B$  are a *complementary pair* of operators,

$$-i[A, B] \geq \text{id}, \quad \text{meaning that } -i\langle [A, B]f, f \rangle \geq \langle f, f \rangle \text{ for all suitable } f.$$

Also, the operators  $A$  and  $B$  are self-adjoint because for suitable functions  $f$  and  $g$ ,

$$\langle Af, g \rangle = \int if' \bar{g} = i \int f' \bar{g} = -i \int f \bar{g}' = \int f \overline{ig'} = \langle f, Ag \rangle,$$

and

$$\langle Bf, g \rangle = \int xf \bar{g} = \int f \bar{xg} = \langle f, Bg \rangle.$$

Now compute with the wavefunction as follows:

$$1 = \langle f, f \rangle \leq -i\langle (AB - BA)f, f \rangle = -i(\langle Bf, Af \rangle - \langle Af, Bf \rangle) = -2 \text{Im}(\langle Af, Bf \rangle).$$

So in fact we have, using the Cauchy-Schwarz inequality for the second step,

$$1 \leq 2|\langle Af, Bf \rangle| \leq 2\|Af\|_2 \cdot \|Bf\|_2 = 2\|f'(x)\|_2 \cdot \|xf(x)\|_2.$$

Recall that the Fourier transform is an isometry that converts differentiation to multiplication by the variable,

$$\|f'(x)\|_2 = \|\widehat{f'}(\xi)\|_2 = 2\pi \|\xi \widehat{f}(\xi)\|_2.$$

Consequently

$$1 \leq 4\pi \|xf(x)\|_2 \cdot \|\xi\widehat{f}(\xi)\|_2.$$

Thus the product of the spreads of  $f$  and  $\widehat{f}$  is bounded from below. Since  $f$  is a wavefunction, its Fourier transform  $\widehat{f}$  is momentum. This is the Heisenberg principle, normalized to the case of functions whose  $A$ -means and  $B$ -means are zero. The general case follows from a change of variables.