THE WEIERSTRASS/HADAMARD FACTORIZATION OF AN ENTIRE FUNCTION

These notes are drawn closely from chapter 5 of **Princeton Lectures in Analysis II: Complex Analysis** by Stein and Shakarchi.

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be nonzero and entire, with infinitely many roots, vanishing to order $m \geq 0$ at 0. The nonzero roots of f, with repetition for multiplicity, form a sequence $\{a_n\}$ such that $\lim_n |a_n| = \infty$. For an initial product form that attempts to factor f, first define

$$E_0(\zeta) = 1 - \zeta,$$

an entire function of ζ that vanishes only for $\zeta = 1$ and goes to 1 as ζ goes to 0. Thus $E_0(z/a_n)$ vanishes only at $z = a_n$, and for fixed z it goes to 1 as n goes to ∞. Then define

$$p_0(z) = z^m \prod_{n=1}^{\infty} E_0(z/a_n) = z^m \prod_{n=1}^{\infty} (1 - z/a_n).$$

However, this product need not even converge, much less converge to an entire function that matches the roots of f. We will see that a sufficient condition for such convergence is that $\sum_{n=1}^{\infty} 1/|a_n|$ converges, but this condition fails unless the a_n are sparse enough.

Recall that $\log(1 - \zeta) = -\sum_{j=1}^{\infty} \frac{\zeta^j}{j}$ (principal branch) for $|\zeta| < 1$, and so exponentiating gives $(1 - \zeta)e^{\sum_{j=1}^{\infty} \frac{\zeta^j}{j}} = 1$ for such ζ. For any nonnegative integer k generalize E_0 to the k-truncation of this expression of 1,

$$E_k(\zeta) = (1 - \zeta)e^{\zeta + \frac{\zeta^2}{2} + \frac{\zeta^3}{3} + \cdots + \frac{\zeta^k}{k}},$$

again an entire function of ζ that vanishes only for $\zeta = 1$. Because

$$E_k(\zeta) = e^{-\sum_{j=k+1}^{\infty} \frac{\zeta^j}{j}} \approx 1 - \tfrac{\zeta^{k+1}}{k+1} \quad \text{for } |\zeta| < 1,$$

$E_k(\zeta)$ goes to 1 more quickly for larger k as ζ goes to 0; this approximation will be made more precise below. Again $E_k(z/a_n)$ vanishes only at $z = a_n$, and so for any nonnegative integer sequence $\{k_n\}_{n \geq 1}$ the expression

$$p_{\{k_n\}}(z) = z^m \prod_{n=1}^{\infty} E_{k_n}(z/a_n) = z^m \prod_{n=1}^{\infty} (1 - z/a_n)e^{z/a_n + \frac{(z/a_n)^2}{2} + \cdots + \frac{(z/a_n)^{k_n}}{k_n}}$$

might be an entire function having the roots as f. This $p_{\{k_n\}}$ improves on p_0 because for large enough n to make z/a_n small, its multiplicands $E_{k_n}(z/a_n)$ can be made as close to 1 as desired by choosing larger k_n, and we will see that in particular the sequence $\{k_n\} = \{n\}$ makes $p_{\{k_n\}}$ converge to an entire function with the same roots as f.

Once we know that some $p_{\{k_n\}}$ is entire with the same roots as f, their quotient $f/p_{\{k_n\}}$ defines an entire function that never vanishes. As will be reviewed, the quotient therefore takes the form e^g with g entire. Thus the factorization of f is

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_n(z/a_n).$$

So far, these ideas are due to Weierstrass. Hadamard added to them, as follows. If f has *finite order*, meaning that for some positive constants A, B, and ρ it satisfies a growth bound

$$|f(z)| \leq Ae^{B|z|^\rho} \quad \text{for all } z,$$

then its roots are sparse; specifically, $\sum_{n=1}^{\infty} |a_n|^{-s}$ converges if $s > \rho$. We will see that in consequence of this, letting $k = \lfloor \rho \rfloor$, the Weierstrass factorization improves to

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n),$$

now with nth multiplicand $E_k(z/a_n)$ rather than $E_n(z/a_n)$. That is, the convergence factors $e^{\sum_{j=1}^{k} \frac{(z/a_n)^j}{j}}$ all have equal length k according to ρ. In practical examples k is often small, e.g., 0 or 1. A second consequence of the sparseness of the roots is that

$$g(z) \text{ is a polynomial of degree at most } k,$$

as we will also see.

Contents

Part 1. Weierstrass Factorization of an Entire Function

1. Estimate of $E_k - 1$

Let k be a nonnegative integer. Recall the definition

$$E_k(\zeta) = (1-\zeta)e^{\zeta+\frac{\zeta^2}{2}+\frac{\zeta^3}{3}+\cdots+\frac{\zeta^k}{k}}.$$

For $k = 0$ we have $E_0(\zeta) = 1 - \zeta$ and so $|E_0(\zeta) - 1| = |\zeta|$ for all $\zeta \in \mathbb{C}$. We generalize this to an estimate of $|E_k(\zeta) - 1|$ for any k, though now with a condition on ζ. The argument will show how the factor $e^{\zeta+\zeta^2/2+\zeta^3/3+\cdots+\zeta^k/k}$ brings $E_k(\zeta)$ closer to 1 for larger k when ζ is small.

Suppose through this paragraph that $|\zeta| \le 1/2$; here the $1/2$ could be any positive $r < 1$ with no essential change to the argument to follow, but we use $1/2$ for definiteness. Then

$$1-\zeta = e^{\log(1-\zeta)} = e^{-\zeta-\frac{\zeta^2}{2}-\frac{\zeta^3}{3}-\cdots-\frac{\zeta^k}{k}-\frac{\zeta^{k+1}}{k+1}-\cdots},$$

and so, because $E_k(\zeta) = (1-\zeta)e^{\zeta+\zeta^2/2+\zeta^3/3+\cdots+\zeta^k/k}$, we have

$$E_k(\zeta) = e^{-\frac{\zeta^{k+1}}{k+1}-\frac{\zeta^{k+2}}{k+2}-\cdots},$$

which certainly goes to 1 as k grows. Loosely, taking the linear approximation of the exponential series and then keeping only its lowest-order term after the constant terms cancel,

$$E_k(\zeta) - 1 \approx 1 + \left(-\tfrac{\zeta^{k+1}}{k+1} - \tfrac{\zeta^{k+2}}{k+2} - \cdots\right) - 1 \approx -\tfrac{\zeta^{k+1}}{k+1}.$$

To make this approximation precise, introduce a convenient abbreviation,

$$E_k(\zeta) = e^w \quad \text{where } w = w_k(\zeta) = -\tfrac{\zeta^{k+1}}{k+1} - \tfrac{\zeta^{k+2}}{k+2} - \cdots.$$

Because $|\zeta| \le 1/2$,

$$|w| \le |\zeta|^{k+1}\sum_{j=0}^{\infty}\frac{1}{2^j} = 2|\zeta|^{k+1},$$

and in particular $|w| \le 1$, even for $k = 0$. This gives $|w|^j \le |w|$ for all $j \ge 1$, and therefore

$$|E_k(\zeta) - 1| = |e^w - 1| \le \sum_{j=1}^{\infty}\frac{|w|^j}{j!} \le (e-1)|w|.$$

Together the previous two displays give our desired estimate, improving the approximation $E_k(\zeta) - 1 \approx -\frac{\zeta^{k+1}}{k+1}$ to a rigorous bound,

$$|E_k(\zeta) - 1| \le 2(e-1)|\zeta|^{k+1} \quad \text{if } |\zeta| \le 1/2. \tag{1}$$

2. Infinite product convergence criterion

Let $\{z_n\}$ be a complex sequence, with $z_n \ne -1$ for all n. We show:

If $\sum_{n=1}^{\infty}|z_n|$ *converges then* $\prod_{n=1}^{\infty}(1+z_n)$ *converges in* $\mathbb{C}^\times$ *and can be rearranged.*

Begin by noting that all but finitely many z_n satisfy $|z_n| \leq 1/2$. We freely work only with these z_n, for which, using the power series of the principal branch $-\pi < \arg(1+z) < \pi$ of $\log(1+z)$ for z in the open unit disk,

$$|\log(1+z_n)| = \left|z_n(1 - z_n/2 + z_n^2/3 + \cdots)\right| \leq 2|z_n|.$$

Thus the sequence $\left\{\sum_{n=1}^N \log(1+z_n)\right\}$ of partial sums of $\sum_{n=1}^\infty \log(1+z_n)$ converges absolutely, and so it converges and can be rearranged. Consequently, because the complex exponential function is continuous, convergence and rearrangeability also hold for the sequence

$$\left\{e^{\sum_{n=1}^N \log(1+z_n)}\right\} = \left\{\prod_{n=1}^N e^{\log(1+z_n)}\right\} = \left\{\prod_{n=1}^N (1+z_n)\right\}.$$

This is the sequence of partial products of $\prod_{n=1}^\infty (1+z_n)$, and the convergence criterion is established. The argument has shown further that $\prod_{n=1}^\infty (1+z_n)$ is nonzero under the hypotheses on $\{z_n\}$, because it is $e^{\sum_{n=1}^\infty \log(1+z_n)}$. The argument has made no claim that $\sum_n \log(1+z_n)$ and $\log\prod_n (1+z_n)$ are equal.

Theorem 2.1. *Let Ω be domain in $\mathbb{C}$. Let $\{\varphi_n\}$ be a sequence of analytic functions on Ω. Suppose that:*

> *For every compact K in Ω*
> *there is a summable sequence $\{x_n\} = \{x_n(K)\}$ in $\mathbb{R}_{\geq 0}$ such that*
> *$|\varphi_n(z)| \leq x_n$ for all n, uniformly over $z \in K$.*

Then the product $p(z) = \prod_{n=1}^\infty (1+\varphi_n(z))$ is analytic on Ω, and its roots are precisely the values $z \in \Omega$ such that $1+\varphi_n(z) = 0$ for some n.

Indeed, the partial products of $p(z)$ are analytic on Ω. For any compact K in Ω the bound $|\varphi_n(z)| \leq x_n$ for all n uniformly over K combines with the argument just given to establish that $p(z)$ converges uniformly on K. Because $p(z)$ on Ω has analytic partial products and converges uniformly on compacta it is analytic, by the Weierstrass theorem. For any $z \in K$ such that $1+\varphi_n(z) \neq 0$ for all n, the argument just given, with $\{\varphi_n(z)\}$ in place of $\{z_n\}$, establishes that $\prod_{n=1}^\infty (1+\varphi_n(z)) \neq 0$.

Example 1. Let a sequence $\{a_n\}$ of nonzero complex numbers be given such that

$$\lim_{n\to\infty} |a_n| = \infty.$$

Let

$$\varphi_n(z) = E_n(z/a_n) - 1 \quad \text{for each } n.$$

Given any compact K in $\mathbb{C}$, there exists $n_o \in \mathbb{Z}_{\geq 0}$ such that $|z/a_n| \leq 1/2$ for all $n \geq n_o$, uniformly over $z \in K$. Let

$$x_n = \begin{cases} \sup_{z\in K} |\varphi_n(z)| & \text{for } n < n_o \\ (e-1)/2^n & \text{for } n \geq n_o. \end{cases}$$

Thus, using (1) from the end of the previous section,

$$|\varphi_n(z)| = |E_n(z/a_n) - 1| \leq 2(e-1)|z/a_n|^{n+1} \leq x_n \quad \text{for all } n \geq n_o \text{ and } z \in K,$$

and certainly $|\varphi_n(z)| \leq x_n$ for all $n < n_o$ and $z \in K$. Because $\{x_n\}$ is summable, this shows that the product $\prod_{n=1}^\infty E_n(z/a_n)$ is entire with roots $\{a_n\}$.

Example 2. Let a sequence $\{a_n\}$ of nonzero complex numbers be given such that

$$\sum_{n=1}^{\infty} |a_n|^{-k-1} \text{ converges for some nonnegative integer } k.$$

This is a stronger hypothesis than in the previous example. Let

$$\varphi_n(z) = E_k(z/a_n) - 1 \quad \text{for each } n,$$

here with E_k rather than E_n as in the previous example. Given any compact K in $\mathbb{C}$, there exists $c > 0$ such that $2(e-1)|z|^{k+1} \le c$ for all $z \in K$, and there exists $n_o \in \mathbb{Z}_{\ge 0}$ such that $|z/a_n| \le 1/2$ for all $n \ge n_o$. Let

$$x_n = \begin{cases} \sup_{z \in K} |\varphi_n(z)| & \text{for } n < n_o \\ c/|a_n|^{k+1} & \text{for } n \ge n_o. \end{cases}$$

Thus, again using (1),

$$|\varphi_n(z)| = |E_k(z/a_n) - 1| \le 2(e-1)|z/a_n|^{k+1} \le x_n \quad \text{for all } n \ge n_o \text{ and } z \in K,$$

and certainly $|\varphi_n(z)| \le x_n$ for all $n < n_o$ and $z \in K$. Because $\{x_n\}$ is summable, this shows that the product $\prod_{n=1}^{\infty} E_k(z/a_n)$ is entire with roots $\{a_n\}$. Especially,

$$\text{if } \sum_{n=1}^{\infty} 1/|a_n| \text{ converges then } \prod_{n=1}^{\infty} (1 - z/a_n) \text{ is entire with roots } \{a_n\},$$
$$\text{if } \sum_{n=1}^{\infty} 1/|a_n|^2 \text{ converges then } \prod_{n=1}^{\infty} (1 - z/a_n)e^{z/a_n} \text{ is entire with roots } \{a_n\}.$$

Example 3. (The Euler–Riemann zeta function; this example is not necessary for the present writeup.) Let Ω be the right half plane $\mathrm{Re}(s) > 1$; the variable name s rather than z is standard in this context. Let

$$\varphi_n(s) = \begin{cases} (1 - p^{-s})^{-1} - 1 = (1 - p^{-s})^{-1} p^{-s} & \text{if } n \text{ is a prime } p \\ 0 & \text{otherwise.} \end{cases}$$

Let K be a compact subset of Ω. There exists some $\sigma > 1$ such that $\mathrm{Re}(s) \ge \sigma$ on K. Let

$$\{x_n\} = \{2n^{-\sigma}\}.$$

For all $n \ge 1$ and $s \in K$, noting that $|1 - p^{-s}| \ge 1 - |p^{-s}| = 1 - p^{-\sigma} \ge 1 - 2^{-1} = 1/2$ and so $|(1 - p^{-s})^{-1}| \le 2$,

$$|\varphi_n(s)| = \begin{cases} |(1 - p^{-s})^{-1} p^{-s}| \le 2p^{-\sigma} = x_n & \text{if } n \text{ is a prime } p \\ 0 \le x_n & \text{otherwise.} \end{cases}$$

Because $\{x_n\}$ is summable, this shows that the product expression $\prod_p (1 - p^{-s})^{-1}$ of the Euler–Riemann zeta function $\zeta(s)$ is analytic and never zero on $\mathrm{Re}(s) > 1$, with no reference to it equaling the sum $\sum_{n=1}^{\infty} n^{-s}$.

3. A non-vanishing analytic function is an exponential

We show: *If Ω is a simply connected region, and if $f : \Omega \longrightarrow \mathbb{C}$ is analytic and never vanishes, then f takes the form e^g for some analytic g on Ω.*

The argument is constructive. Let a be a point of Ω, and take any value of $\log(f(a))$. Introduce

$$g(z) = \log(f(a)) + \int_{\zeta=a}^{z} \frac{f'(\zeta)\,\mathrm{d}\zeta}{f(\zeta)},$$

well defined because Ω is simply connected. Then $g'(z) = f'(z)/f(z)$, and so

$$\big(f(z)e^{-g(z)}\big)' = (f'(z) - f(z) \cdot f'(z)/f(z))e^{-g(z)} = 0.$$

Also $f(a)e^{-g(a)} = 1$, and therefore $f = e^g$.

Especially, if the product $p(z) = z^m \prod_n E_{k_n}(z/a_n)$ is entire and has the same roots as $f(z)$, then $f(z) = e^{g(z)}p(z)$ for some entire g.

4. Weierstrass product

Let f be nonzero entire and have nonzero roots $\{a_n\}$. These roots satisfy the condition $\lim_n |a_n| = \infty$, and so the first example at the end of section 2 shows that the product $p(z) = z^m \prod_{n=1}^{\infty} E_n(z/a_n)$ converges to an entire function having the same roots as f. Section 3 therefore gives the Weierstrass factorization of f,

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_n(z/a_n).$$

Here the convergence factor of E_n gets longer as n grows, and all that we know about g is that it is entire.

Part 2. Hadamard Factorization of a Finite-Order Entire Function

Let f be a nonzero entire function of finite order at most $\rho > 0$, meaning that for some positive constants A and B it satisfies a growth bound

$$|f(z)| \le Ae^{B|z|^\rho} \quad \text{for all } z.$$

Here the condition *for all z* can be replaced by *for all z such that $|z| > R$ for some R.* The actual order of f is the infimum of all such ρ; for example, if $|f(z)| \le Ae^{|z|\ln|z|}$ but $|f(z)| \not\le Ae^{|z|}$, or if $|f(z)| \le p(|z|)e^{|z|}$ for some polynomial p but $|f(z)| \not\le Ae^{|z|}$, then still f has order 1. If f has finite order ρ_f and similarly for g then fg has finite order $\max\{\rho_f, \rho_g\}$.

Let f have order $m \in \mathbb{Z}_{\ge 0}$ at 0. Let $\{a_n\}$ be the nonzero roots of f, with multiplicity, so that $|a_n| \to \infty$. For any $r \ge 0$, let $\mathfrak{n}(r) = \mathfrak{n}_f(r)$ denote the number of nonzero roots a_n of f such that $|a_n| < r$. The terminology f, ρ, m, $\{a_n\}$, $\mathfrak{n}$ is in effect for the rest of this writeup. We note that if f is entire with a root of order m at 0, then f has order at most ρ if and only if f/z^m has order at most ρ.

5. Sparseness of roots: statement

To prepare for Hadamard's factorization theorem, our first main goal is as follows.

Theorem 5.1. *Let f, ρ, $\{a_n\}$, and $\mathfrak{n}$ be as just above. Then*

1. $\mathfrak{n}(r) \le C|r|^\rho$ *for all large enough r.*
2. $\sum_{n=1}^{\infty} |a_n|^{-s}$ *converges for all $s > \rho$.*

The main result needed to prove the theorem is a variant of Jensen's formula, to be established next.

6. Jensen's formula

For $R > 0$ and φ analytic on the closed complex ball $\overline{B}_R$, where $\varphi(0) \neq 0$ and $\varphi \neq 0$ on the boundary circle C_R, letting the finitely many roots of φ be denoted $\{a_n\}$ with repetition for multiplicity,

$$\ln|\varphi(0)| = \sum_n \ln\frac{|a_n|}{R} + \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \ln|\varphi(Re^{i\theta})|\,\mathrm{d}\theta. \tag{J1}$$

The proof begins with two reductions:

- The formula for general R follows from the formula for $R = 1$.
- The formula for a product $\varphi_1\varphi_2$ follows from the formula for φ_1 and for φ_2.
- The decomposition $\varphi(z) = \varphi_o(z)\prod_n(z - a_n)$, where $\varphi_o(z)$ is the analytic extension of $\varphi(z)/\prod_n(z-a_n)$, reduces the formula for $R = 1$ to two cases, where φ has no roots and where $\varphi(z) = z - a_1$.

If φ on $\overline{B}_1$ has no roots then it takes the form $\varphi = e^g$, as discussed above. Let $g = u + iv$ with u and v harmonic conjugates, so that $|\varphi| = e^u$ and thus $\ln|\varphi| = u$. The mean value property of harmonic functions gives

$$\ln|\varphi(0)| = u(0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(e^{i\theta})\,\mathrm{d}\theta = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \ln|\varphi(e^{i\theta})|\,\mathrm{d}\theta.$$

If $\varphi(z) = z - a_1$ with $|a_1| < 1$ then the desired formula reduces to

$$\int_{\theta=0}^{2\pi} \ln|e^{i\theta} - a_1|\,\mathrm{d}\theta = 0.$$

Because $\ln|e^{i\theta} - a_1| = \ln|1 - e^{-i\theta}a_1|$, and then we may replace θ by $-\theta$ in the integral, this is

$$\int_{\theta=0}^{2\pi} \ln|1 - a_1e^{-i\theta}|\,\mathrm{d}\theta = 0.$$

Similarly to the first case, the function $f(z) = 1 - a_1 z$ takes the form e^g on $\overline{B}_1$, where $g = u + iv$, and so again the integral is a mean value integral for u. But this time $u(0) = 0$ because $\varphi(0) = 1$, and so the integral is 0 as desired.

A variant of Jensen's formula is as follows.

$$\ln|\varphi(0)| = -\int_{x=0}^{R} \mathfrak{n}_\varphi(x)\frac{\mathrm{d}x}{x} + \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\ln|\varphi(Re^{i\theta})|\,\mathrm{d}\theta. \tag{J2}$$

This follows from Jensen's formula (J1) if we can establish the equality

$$-\int_{x=0}^{R}\mathfrak{n}(x)\frac{\mathrm{d}x}{x} = \sum_n \ln\frac{|a_n|}{R}\,,$$

in which $\mathfrak{n} = \mathfrak{n}_\varphi$. This equality reduces to the case $R = 1$. Define $\eta_n(x)$ to be 1 if $x > |a_n|$ and 0 otherwise, so that $\mathfrak{n}(x) = \sum_n \eta_n(x)$, and compute,

$$-\int_{x=0}^{1}\mathfrak{n}(x)\frac{\mathrm{d}x}{x} = -\sum_n\int_{x=0}^{1}\eta_n(x)\frac{\mathrm{d}x}{x} = -\sum_n\int_{x=|a_n|}^{1}\frac{\mathrm{d}x}{x} = \sum_n \ln|a_n|.$$

7. Sparseness of roots: proof

We prove part (1) of Theorem 5.1. Partially reiterating the theorem's hypotheses, the nonzero entire function f has finite order at most ρ and root-counting function $\mathfrak{n}$, and we want to show that

$$\mathfrak{n}(r) \leq Cr^\rho \quad \text{for some } C \in \mathbb{R}_{>0} \text{ and all large enough } r.$$

It suffices to prove this in the case $f(0) \neq 0$. For any $r \in \mathbb{R}_{>0}$, let $R = 2r$, so that $\int_r^R \mathrm{d}x/x = \ln 2$. Then, using the variant Jensen's formula (J2) for the last step in the next computation,

$$\mathfrak{n}(r)\ln 2 = \mathfrak{n}(r)\int_r^R \frac{\mathrm{d}x}{x} \leq \int_0^R \mathfrak{n}(x)\frac{\mathrm{d}x}{x} = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \ln|f(Re^{i\theta})|\,\mathrm{d}\theta - \ln|f(0)|.$$

Consequently,

$$\mathfrak{n}(r) \leq C_1 r^\rho + C_2 \quad \text{for some } C_1 \in \mathbb{R}_{>0} \text{ and } C_2 \in \mathbb{R}, \text{ for all } r \in \mathbb{R}_{>0},$$

and the result follows.

We prove part (2) of Theorem 5.1. Recall that the nonzero roots of f are $\{a_n\}$. We show that $\sum_n |a_n|^{-s}$ converges if $s > \rho$. Indeed, we now have $\mathfrak{n}(r) \leq Cr^\rho$ for all $r \geq 2^{j_o}$ for some nonnegative integer j_o. Compute,

$$\sum_{|a_n| \geq 2^{j_o}} |a_n|^{-s} = \sum_{j=j_o}^{\infty} \sum_{2^j \leq |a_n| < 2^{j+1}} |a_n|^{-s} \leq \sum_{j=j_o}^{\infty} \mathfrak{n}(2^{j+1})2^{-js} \leq C\sum_{j=j_o}^{\infty} 2^{(j+1)\rho - js}.$$

The last sum is $2^\rho \sum_{j=j_o}^{\infty}(2^{\rho-s})^j$, which converges because $s > \rho$.

8. Hadamard product, part 1

Let f be nonzero entire of finite order at most $\rho > 0$. Consider the nonnegative integer

$$k = \lfloor \rho \rfloor,$$

so that $k \leq \rho < k+1$. As just shown, the nonzero roots $\{a_n\}$ are such that $\sum_{n=1}^{\infty} |a_n|^{-k-1}$ converges, and so the second example at the end of section 2 shows that the product $z^m \prod_{n=1}^{\infty} E_k(z/a_n)$ converges to an entire function having the same roots as f. Section 3 therefore gives the Hadamard factorization of f,

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n).$$

Here all the terms $E_k(z/a_n)$ have convergence factors of the same length. The remaining work is to analyze $g(z)$. This is more technical.

9. Lower bound

Freely ignoring any root of f at 0, to show that g is a low degree polynomial we must bound the quotient $f(z)/\prod_{n=1}^{\infty} E_k(z/a_n)$ from above, and this requires bounding the product $\prod_{n=1}^{\infty} E_k(z/a_n)$ from below.

Again with f having finite order at most ρ and with $k = \lfloor \rho \rfloor$, consider any s such that $\rho < s < k+1$. Thus $s > k$. Consider any $z \in \mathbb{C}$. We want to show that subject to a condition on z to be specified, $\prod_{n=1}^{\infty} E_k(z/a_n)$ is bounded from below as follows,

$$\left|\prod_{n=1}^{\infty} E_k(z/a_n)\right| \geq e^{-c|z|^s}.$$

For the infinitely many values n such that $|z/a_n| \leq 1/2$, we have shown in section 1 that $E_k(z/a_n) = e^w$ where $w = -\sum_{j\geq k+1}(z/a_n)^j/j$ and so $|w| \leq 2|z/a_n|^{k+1}$. Because $|e^w| \geq e^{-|w|}$,

$$|E_k(z/a_n)| \geq e^{-2|z/a_n|^{k+1}} = e^{-2|z/a_n|^{k+1-s}|z/a_n|^s} \geq e^{-(1/2)^{k-s}|z|^s/|a_n|^s}.$$

Thus, because $\sum_{n=1}^{\infty}|a_n|^{-s}$ converges,

$$\left|\prod_{n:|z/a_n|\leq 1/2} E_k(z/a_n)\right| \geq e^{-c|z|^s},$$

with $c = 2^{s-k}\sum_{n=1}^{\infty}|a_n|^{-s}$.

For the finite many values n such that $|z/a_n| > 1/2$,

$$|E_k(z/a_n)| = |1 - z/a_n|\,|e^{\sum_{j=1}^{k}(z/a_n)^j/j}|,$$

and, again because $|e^w| \geq e^{-|w|}$, and noting that $|2z/a_n| \geq 1$, the exponential term satisfies

$$|e^{\sum_{j=1}^{k}(z/a_n)^j/j}| \geq e^{-\sum_{j=1}^{k}|2z/a_n|^j/(2^j j)|} \geq e^{-c|z|^k} \geq e^{-c|z|^s},$$

with $c = k2^k/a_1^k$. So in order to show the condition $|\prod_{n=1}^{\infty} E_k(z/a_n)| \geq e^{-c|z|^s}$, only the non-exponential terms remain, and we need to show that

$$\prod_{n:|z/a_n|>1/2} |1 - z/a_n| \geq e^{-c|z|^s}.$$

However, this is not guaranteed until we add a condition on z. For each positive integer n, let B_n denote the open ball about a_n of radius $|a_n|^{-k-1}$. We stipulate that z lie outside $\bigcup_n B_n$. For such z,

$$|1 - z/a_n| = |z - a_n|/|a_n| \geq |a_n|^{-k-2} \geq (2|z|)^{-k-2}.$$

Take $\varepsilon > 0$ such that $s - \varepsilon > \rho$, and thus $\mathfrak{n}(2|z|) \leq c|z|^{s-\varepsilon}$ for large z. Thus,

$$\prod_{n:|z/a_n|>1/2} |1 - z/a_n| \geq (2|z|)^{-(k+2)\mathfrak{n}(2|z|)} \geq (2|z|)^{-c|z|^{s-\varepsilon}},$$

and the desired result follows,

$$\prod_{n:|z/a_n|>1/2} |1 - z/a_n| \geq e^{-c|z|^{s-\varepsilon}\ln(2|z|)} \geq e^{-c|z|^s}.$$

For each positive integer n, again let B_n denote the open ball about a_n of radius $|a_n|^{-k-1}$, let A_n denote the open annulus generated by rotating B_n around 0, and let I_n denote the intersection of A_n with $\mathbb{R}_{>0}$. For all large integers N, the interval $[N, N+1)$ contains a point r disjoint from $\bigcup_n I_n$, and so the circle C_r is disjoint from $\bigcup_n A_n$, therefore disjoint from $\bigcup_n B_n$. Thus there is a sequence of positive values r that goes to ∞ such that each circle C_r is disjoint from $\bigcup_n B_n$.

10. An entire function with polynomial-growth real part is a polynomial

We show: *Let $g = u + iv$ be entire and satisfy $u(re^{i\theta}) \le Cr^s$ for a sequence of positive values r that goes to ∞, with $s \ge 0$. Then g is a polynomial of degree at most s.*

Because u is bounded only from one side, as compared to a bound on $|u|$, much less on $|g|$, the proof is more than simply Cauchy's bound. Take any r as just described and any integer $n > s$. Cauchy's formula gives

$$\frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{\theta=0}^{2\pi} \frac{g(re^{i\theta})}{(re^{i\theta})^{n+1}}\,\mathrm{d}(re^{i\theta}),$$

which is to say,

$$\frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi r^n}\int_{\theta=0}^{2\pi} g(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta.$$

Also, Cauchy's theorem gives $\int_{\theta=0}^{2\pi} g(re^{i\theta})e^{i(n-1)\theta}\,\mathrm{d}(re^{i\theta}) = 0$, and it follows that $\int_{\theta=0}^{2\pi} g(re^{i\theta})e^{in\theta}\,\mathrm{d}\theta = 0$, from which by complex conjugation,

$$0 = \frac{1}{2\pi r^n}\int_{\theta=0}^{2\pi} \overline{g}(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta.$$

The previous two displayed equations combine to give, recalling that $g = u + iv$ and so $g + \overline{g} = 2u$,

$$\frac{g^{(n)}(0)}{n!} = \frac{1}{\pi r^n}\int_{\theta=0}^{2\pi} u(re^{i\theta})e^{-in\theta}\,\mathrm{d}\theta,$$

or, recalling that $u(re^{i\theta}) \le Cr^s$ and noting that because Cr^s is independent of θ and $\int_{\theta=0}^{2\pi} e^{-in\theta}\,\mathrm{d}\theta = 0$,

$$-\frac{g^{(n)}(0)}{n!} = \frac{1}{\pi r^n}\int_{\theta=0}^{2\pi} (Cr^s - u(re^{i\theta}))e^{-in\theta}\,\mathrm{d}\theta,$$

from which, because $Cr^s - u(re^{i\theta}) \ge 0$ for all θ,

$$\frac{|g^{(n)}(0)|}{n!} \le \frac{1}{\pi r^n}\int_{\theta=0}^{2\pi} (Cr^s - u(re^{i\theta}))\,\mathrm{d}\theta = 2Cr^{s-n} - 2u(0)r^{-n}.$$

Let r grow to show that $g^{(n)}(0) = 0$. Thus the entire function

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n \quad \text{for all } z \in \mathbb{C}$$

is a polynomial of degree at most s, as claimed.

11. Hadamard product, part 2

Our nonzero entire function f has finite order at most ρ, has a root of order $m \ge 0$ at 0, and has nonzero roots $\{a_n\}$. As before, let

$$k = \lfloor \rho \rfloor,$$

and consider any s such that

$$\rho < s < k + 1.$$

Already we have

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_k(z/a_n).$$

Now we show that g is a polynomial of degree at most k.

For a sequence of positive values r that goes to ∞, we have

$$\left| \prod_{n=1}^{\infty} E_k(z/a_n) \right| \geq e^{-c|z|^s} \quad \text{if } |z| = r,$$

from which certainly

$$\left| z^m \prod_{n=1}^{\infty} E_k(z/a_n) \right| \geq e^{-c|z|^s} \quad \text{if } |z| = r.$$

Consequently, with $g = u + iv$, because also $|f(z)| \leq Ae^{B|z|^\rho}$,

$$e^{u(z)} = |e^{g(z)}| \leq Ae^{B|z|^\rho + c|z|^s} \leq e^{C|z|^s} \quad \text{if } |z| = r,$$

which is to say,

$$u(re^{i\theta}) \leq Cr^s.$$

As just shown, $g(z)$ is a polynomial of degree at most s, hence degree at most $\lfloor s \rfloor$, which is to say degree at most k.

Part 3. **Examples**

12. The Euler–Riemann zeta function

We establish Hadamard's product formula

$$(s-1)\zeta(s) = e^{a+bs} \prod_{n \geq 1} \left(1 + \frac{s}{2n}\right) e^{-s/2n} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \quad s \in \mathbb{C}.$$

Here ρ runs through the nontrivial zeros of the zeta function, those lying in the critical strip $0 < \mathrm{Re}(s) < 1$. Although the values of a and b aren't particularly important, they are $a = -\log 2$ and $b = \zeta'(0)/\zeta(0) - 1 = \log 2\pi - 1$.

The function

$$Z_{\text{entire}}(s) = s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad s \in \mathbb{C}$$

extends from an analytic function on the right half plane $\mathrm{Re}(s) > 1$ to an entire function, and the extension is symmetric about the vertical line $\mathrm{Re}(s) = 1/2$, i.e., it is invariant under the replacing s by $1 - s$.

Let $s = \sigma + it$. For $\sigma \geq 1/2$, we have upper bounds of the four constituents s, $\pi^{-s/2}$, $\Gamma(s)$, and $(1-s)\zeta(s)$ of $Z_{\text{entire}}(s)$, as follows:

- $|s| \leq e^{|s|}$ for large s.
- $|\pi^{-s/2}| = \pi^{-\sigma/2} \leq \pi^{-1/4}$.
- $|\Gamma(s/2)| \leq \Gamma(\sigma/2)$, and by Stirling's formula this is asymptotically at most $Ae^{\sigma \ln \sigma}$, in turn at most $Ae^{|s| \ln |s|}$.
- Some analysis shows that after extending $\zeta(s) - 1/(s-1)$ leftward from $\sigma > 1$ to $\sigma > 0$, we have $|\zeta(s) - 1/(s-1)| \leq \zeta(3/2)|s|$ for $\sigma \geq 1/2$, and so $|(s-1)\zeta(s)| \leq 1 + \zeta(3/2)|s(s-1)| \leq 1 + \zeta(3/2)|s|(|s|+1)$ for $\sigma \geq 1/2$; from this, certainly $|(1-s)\zeta(s)| \leq e^{|s|}$ for large s with $\mathrm{Re}(s) \geq 1/2$.

Altogether these give the upper bound

$$|Z_{\text{entire}}(s)| \le Ae^{B|s|\ln|s|}, \quad \text{Re}(s) \ge 1/2.$$

And because $|1-s| \sim |s|$, the symmetry of $Z_{\text{entire}}(s)$ gives

$$|Z_{\text{entire}}(s)| \le Ae^{B|s|\ln|s|}, \quad \text{Re}(s) < 1/2.$$

Altogether $Z_{\text{entire}}(s)$ has order at most 1, and therefore it has a Hadamard product expansion

$$s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s) = e^{a+bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \quad s \in \mathbb{C}.$$

But also the reciprocal gamma function has a well known product expansion, in which γ denotes the Euler-Mascheroni constant,

$$1/\Gamma(s) = e^{\gamma s}s\prod_{n\ge 1}\left(1+\frac{s}{n}\right)e^{-s/n}, \quad s \in \mathbb{C}.$$

Such a product expression, though with $e^{a'+b's}$ rather than $e^{\gamma s}$, follows from the estimate $|1/\Gamma(s)| \le Ae^{B|s|\ln|s|}$ (see Stein and Shakarchi, Theorem 6.1.6, page 165). Divide the penultimate display by $-s\pi^{-s/2}\Gamma(s/2)$ and use the previous display to get, with new a and b, the claimed result,

$$(s-1)\zeta(s) = e^{a+bs}\prod_{n\ge 1}\left(1+\frac{s}{2n}\right)e^{-s/2n}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \quad s \in \mathbb{C}.$$

13. The sine function

One readily shows that the sine function has order 1, and so for some $b \in \mathbb{C}$,

$$\sin(\pi z) = e^{bz}\pi z\prod_{n\ge 1}\left(1-\frac{z^2}{n^2}\right).$$

We show that $b = 0$. Indeed, write the previous display as

$$\frac{\sin(\pi z)}{\pi z} = e^{bz}\prod_{n\ge 1}\left(1-\frac{z^2}{n^2}\right),$$

with the left side continued analytically to 1 at $z = 0$. This says that for small z,

$$1 + o(z) = \big(1 + bz + o(z)\big)\big(1 + o(z)\big) = 1 + bz + o(z),$$

from which $b = 0$. As an exercise, tracking z^2-terms as well shows that $\zeta(2) = \pi^2/6$. In fact, an elementary formula for $\zeta(2d)$ where $d = 1, 2, 3, \ldots$ can be extracted from the Taylor series expansion and the product expansion of $\sin(\pi z)/(\pi z)$. This is unsurprising in light of a well known method to obtain $\zeta(2d)$ from the sum expansion of $\pi\cot(\pi z)$, the logarithmic derivative of $\sin(\pi z)$.