SYMMETRY OF THE GAMMA FUNCTION

Extending work of Euler from a century earlier, Riemann showed that the completed zeta function,

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1,$$

has a meromorphic continuation to the full s-plane, with a simple pole of residue 1 at $s = 1$, and the continuation is invariant under the complex transformation $s \mapsto 1 - s$. This extends ζ itself from $\mathrm{Re}(s) > 1$ to all s, provided that Γ extends from $\mathrm{Re}(s) > 0$ to all s. The extension of Γ follows from the identity

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < \mathrm{Re}(s) < 1.}$$

That is, the extended Γ is defined by stipulating that the equality in the previous display hold for all s. Especially, for s near a nonnegative integer $-n$,

$$\Gamma(s) = \Gamma(s + n - n) = \frac{\pi}{\Gamma(1-s)\sin(\pi(s+n-n))} \sim \frac{(-1)^n}{n!(s+n)},$$

showing that Γ has a simple pole of residue $(-1)^n/n!$ at $-n$. With these ideas in mind, this writeup establishes the boxed identity above.

The *Haar measure* of the multiplicative group of positive real numbers $(\mathbb{R}^\times_{>0}, \cdot)$ is

$$\mathrm{d}\mu(t) = \frac{\mathrm{d}t}{t}\,.$$

Compatibly with the familiar rule $\mathrm{d}(t + c) = \mathrm{d}t$ and $\mathrm{d}(at) = a\,\mathrm{d}t$ for the usual measure $\mathrm{d}t$ of the additive group $(\mathbb{R}, +)$, we have

$$\mathrm{d}\mu(ct) = \frac{\mathrm{d}(ct)}{ct} = \frac{\mathrm{d}t}{t} = \mathrm{d}\mu(t).$$

and

$$\mathrm{d}\mu(t^a) = \frac{\mathrm{d}(t^a)}{t^a} = a\frac{\mathrm{d}t}{t} = a\,\mathrm{d}\mu(t).$$

The integral $\int_{t=1}^{\infty} t^s\,\mathrm{d}\mu(t)$ converges for $\mathrm{Re}(s) < 0$, and so, because $\mathrm{d}\mu(t^{-1}) = -\mathrm{d}\mu(t)$, the integral $\int_{t=0}^{1} t^s\,\mathrm{d}\mu(t)$ converges for $\mathrm{Re}(s) > 0$.

The *gamma function* is defined as an integral,

$$\Gamma(s) = \int_{\mathbb{R}^\times_{>0}} t^s e^{-t}\,\mathrm{d}\mu(t), \quad \mathrm{Re}(s) > 0.$$

The usual notation for the gamma integral is

$$\Gamma(s) = \int_{t=0}^{\infty} t^s e^{-t}\,\mathrm{d}t/t, \quad \mathrm{Re}(s) > 0,$$

but here it should be understood that the lower limit of integration 0 is just as improper as the upper limit ∞. Despite the lower limit of integration being improper, the integral converges for $\mathrm{Re}(s) > 0$, as just explained. Also, the gamma integral converges at its improper upper limit of integration because the exponential decay of e^{-t} dominates the polynomial growth of t^s.

The basic properties of gamma are

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.
- $\Gamma(s+1) = s\Gamma(s)$, so that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$.

The volume of the n-dimensional unit ball is $\pi^{n/2}/(n/2)!$ for $n = 1, 2, 3, \cdots$, where naturally $(n/2)!$ is understood to mean $\Gamma(n/2+1)$.

Now we establish the desired identity,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < \mathrm{Re}(s) < 1.$$

To do so, it suffices to consider only *real* s between 0 and 1. For such s, the definition of gamma gives

$$\Gamma(s)\Gamma(1-s) = \iint_{\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}} w^s x^{1-s} e^{-w} e^{-x} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(w).$$

Replace x by wx and recall that $\mathrm{d}\mu(wx) = \mathrm{d}\mu(x)$,

$$\Gamma(s)\Gamma(1-s) = \iint_{\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}} w x^{1-s} e^{-w(1+x)} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(w).$$

Replace w by $w/(1+x)$, again with no effect on the Haar measure, to get

$$\begin{aligned} \Gamma(s)\Gamma(1-s) &= \iint_{\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}} w \frac{x^{1-s}}{1+x} e^{-w} \, \mathrm{d}\mu(w) \, \mathrm{d}\mu(x) \\ &= \int_{w=0}^{\infty} e^{-w} \, \mathrm{d}w \int_{x=0}^{\infty} \frac{x^{-s}}{1+x} \, \mathrm{d}x. \end{aligned}$$

The w-integral is 1. So altogether,

$$\Gamma(s)\Gamma(1-s) = \int_{x=0}^{\infty} \frac{x^{-s}}{1+x} \, \mathrm{d}x.$$

And we have evaluated this last integral by contour integration,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < s < 1.$$

As mentioned above, the result extends by uniqueness to all complex s such that $0 < \mathrm{Re}(s) < 1$, and it extends Γ to all of $\mathbb{C}$.