A GAMMA FUNCTION IDENTITY

The *Haar measure* of the multiplicative group of positive real numbers $(\mathbf{R}^\times_{>0}, \cdot)$ is

$$d\mu(t) = \frac{dt}{t}\,.$$

Compatibly with the familiar rule $d(t+c) = dt$ for the usual measure dt of the additive group $(\mathbf{R}, +)$, we have

$$d\mu(ct) = \frac{d(ct)}{ct} = \frac{dt}{t} = d\mu(t).$$

The *gamma function* is

$$\Gamma(s) = \int_{\mathbf{R}^\times_{>0}} t^s e^{-t}\, d\mu(t), \quad \mathrm{Re}(s) > 0.$$

The usual notation for the gamma integral is

$$\Gamma(s) = \int_{t=0}^{\infty} t^s e^{-t}\, dt/t, \quad \mathrm{Re}(s) > 0,$$

but here it should be understood that the lower limit of integration 0 is just as improper as the upper limit ∞. Recall that for $t > 0$,

$$|t^s| = t^{\mathrm{Re}(s)}.$$

Thus the gamma integral converges at its improper lower limit of integration since the integrand is $\mathcal{O}(t^{\mathrm{Re}(s)-1})$. Also, the gamma integral converges at its improper upper limit of integration because the exponential decay of e^{-t} dominates the polynomial growth of t^{s-1}.

The basic properties of gamma are

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.
- $\Gamma(s+1) = s\Gamma(s)$, so that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$.

The volume of the n-dimensional unit ball is $\pi^{n/2}/(n/2)!$ for $n = 1, 2, 3, \cdots$, where naturally $(n/2)!$ is understood to mean $\Gamma(n/2+1)$.

A standard gamma identity is

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < \mathrm{Re}(s) < 1.}$$

To establish the identity, it suffices to consider *real* s between 0 and 1. For such s, the definition of gamma gives

$$\Gamma(s)\Gamma(1-s) = \int_{\mathbf{R}^\times_{>0}} \int_{\mathbf{R}^\times_{>0}} u^s v^{1-s} e^{-u} e^{-v}\, d\mu(v)\, d\mu(u).$$

Replace v by uv in the inner integral and recall that $d\mu(uv) = d\mu(v)$,

$$\Gamma(s)\Gamma(1-s) = \int_{\mathbf{R}^\times_{>0}} \int_{\mathbf{R}^\times_{>0}} u v^{1-s} e^{-u(1+v)}\, d\mu(v)\, d\mu(u).$$

Exchange the order of integration, replace u by $u/(1+v)$ in the inner integral (again with no effect on the Haar measure), and re-exchange the order to get

$$\begin{aligned}\Gamma(s)\Gamma(1-s) &= \int_{\mathbf{R}_{>0}^{\times}} \int_{\mathbf{R}_{>0}^{\times}} uv^{1-s}e^{-u(1+v)}\,d\mu(u)\,d\mu(v) \\ &= \int_{\mathbf{R}_{>0}^{\times}} \int_{\mathbf{R}_{>0}^{\times}} \frac{u}{1+v}v^{1-s}e^{-u}\,d\mu(u)\,d\mu(v) \\ &= \int_{\mathbf{R}_{>0}^{\times}} \frac{v^{1-s}}{1+v} \int_{\mathbf{R}_{>0}^{\times}} ue^{-u}\,d\mu(u)\,d\mu(v).\end{aligned}$$

The inner integral is $\Gamma(1) = 1$. In sum,

$$\Gamma(s)\Gamma(1-s) = \int_{\mathbf{R}_{>0}^{\times}} \frac{v^{1-s}}{1+v}\,d\mu(v) = \int_{v=0}^{\infty} \frac{v^{-s}}{1+v}\,dv.$$

And we have evaluated this last integral by contour integration,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < s < 1.$$

As mentioned above, the result extends by uniqueness to all complex s such that $0 < \mathrm{Re}(s) < 1$.