SUFFICIENT CONDITIONS FOR f' AT A POINT

Let Ω be a region in $\mathbb{C}$, and let $f : \Omega \longrightarrow \mathbb{C}$ be a function. Write $f = u + iv$, and write $z = x + iy$ for points z of Ω. For the derivative $f'(z)$ to exist a point z of Ω it is necessary but not sufficient that the Cauchy–Riemann equations hold at z.

Existence Theorem. *Let z be a point of Ω. The following conditions are sufficient for the existence of $f'(z)$.*

- *u_x, u_y, v_x, and v_y exist at z and on a ball about z,*
- *u_x, u_y, v_x, and v_y are continuous at z,*
- *The Cauchy–Riemann equations hold at z.*

These conditions can be weakened with a little effort (i.e., keeping track of just how much of their strength is required in the following proof), but doing so is silly because in practice they are all we need. We are interested in pointwise differentiability over the whole region, not differentiability at just one point of the region, and eventually we will show that differentiability over the region implies the conditions in the existence theorem (and considerably more) at each point. A weaker-but-fussier list of conditions serves no purpose other than being more irritating to verify.

Proof. As usual, treat u and v as functions of the two real variables x and y rather than of the complex variable z. To show that $f'(z)$ exists and that it is equal to $(u_x + iv_x)(x, y)$, it suffices to show that

$$f(z + \Delta z) - f(z) = \Delta z \cdot (u_x + iv_x)(x, y) + o(\Delta z).$$

So compute:

$$\begin{aligned} u(x + \Delta x,&y + \Delta y) - u(x, y) \\ &= u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) + u(x, y + \Delta y) - u(x, y) \\ &= \Delta x \cdot u_x(x, y + \Delta y) + o(\Delta x) + \Delta y \cdot u_y(x, y) + o(\Delta y). \end{aligned}$$

This calculation has used the existence of u_x *near* (x, y), not only at (x, y). Also, the continuity of u_x at (x, y) gives the relation

$$u_x(x, y + \Delta y) = u_x(x, y) + o_y(1) \quad \text{where } o_y(1) \to 0 \text{ as } \Delta y \to 0.$$

Note that $\Delta x \cdot o_y(1)$ is $o(\Delta z)$. Therefore,

$$\begin{aligned} u(x + \Delta x,&y + \Delta y) - u(x, y) \\ &= \Delta x \cdot u_x(x, y) + o(\Delta z) + o(\Delta x) + \Delta y \cdot u_y(x, y) + o(\Delta y) \\ &= \Delta x \cdot u_x(x, y) + \Delta y \cdot u_y(x, y) + o(\Delta z). \end{aligned}$$

Similarly,

$$iv(x + \Delta x, y + \Delta y) - iv(x, y) = \Delta x \cdot iv_x(x, y) + i\Delta y \cdot v_y(x, y) + o(\Delta z).$$

Thus altogether,

$$f(z + \Delta z) - f(z) = \Delta x \cdot (u_x + iv_x)(x, y) + i\Delta y \cdot (v_y - iu_y)(x, y) + o(\Delta z).$$

Apply the Cauchy–Riemann equations at (x, y) to get

$$f(z + \Delta z) - f(z) = (\Delta x + i\Delta y) \cdot (u_x + iv_x)(x, y) + o(\Delta z).$$

That is,

$$f(z + \Delta z) - f(z) = \Delta z \cdot (u_x + iv_x)(x, y) + o(\Delta z).$$

This is precisely the desired differentiability of f at z. □