COMPLEX DIFFERENTIABILITY AND VECTOR DIFFERENTIABILITY

Let $\Omega \subset \mathbf{C}$ be a region, and consider a map

$$f : \Omega \to \mathbf{C}.$$

We may also view Ω as a subset of $\mathbf{R}^2$ and f as a vector-valued function

$$F = (u, v) : \Omega \to \mathbf{R}^2.$$

The result to be shown here is that *if the complex-valued function f of a complex variable z is differentiable, then so is the vector-valued function $F = (u, v)$ of a vector variable (x, y), in the naturally compatible way.* The converse statement requires the Cauchy–Riemann equations.

Why show this? Two traditional answers are:

- For peace of mind. With the notion of derivative now redundantly defined, it is worth checking that the definitions are compatible.
- For practice. Comparing the two definitions of differentiability is a good opportunity to rehearse each of them.

However, while these answers are plausible, I find them only partially convincing. The countless people who have made use of both definitions are neither dupes nor conspirators, and so while the first motivation is virtuous, it also has a tinge of obsessive compulsion or even paranoia. As for the second, there are endless opportunities for self-drill, and so we need to choose among them judiciously. More importantly, even if these answers impel us to solve the problem, *they don't tell us what kind of solution we would like.* My answer is:

- *Because it should be easy.* The main idea of beginning complex analysis is that *complex differentiability has significant consequences.* Since complex differentiability is such a strong condition, it should entail vector differentiability immediately, and so the real exercise is to make the solution *transparent* and then reflect on *why* we were able to do so.

Recall from the definition of vector derivative from multivariable calculus:

For any point $(x, y) \in \Omega$, the derivative $DF_{(x,y)}$ is defined as the unique linear map $L : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{|F(x+h, y+k) - F(x, y) - L(h, k)|}{|(h, k)|} = 0, \tag{1}$$

if such a linear map L exists.

From calculus, if $DF_{(x,y)}$ exists then its matrix is

$$F'(x, y) = \begin{bmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{bmatrix}.$$

(Also from calculus, if the component functions u and v are $\mathcal{C}^{(1)}$ in some neighborhood of (x, y) then $DF_{(x,y)}$ does indeed exist. But while these sufficient conditions are perfectly adequate in practice, they are not necessary and they will play no role in the argument to follow.)

Suppose that $f'(z)$ exists at the point $z = x + iy$. Then $DF_{(x,y)}$ also exists, and its matrix is

$$M = \begin{bmatrix} u_x(x,y) & -v_x(x,y) \\ v_x(x,y) & u_x(x,y) \end{bmatrix}.$$

Proof. By the Cauchy–Riemann equations, if the vector derivative $DF_{(x,y)}$ exists then it must be the linear map L with the displayed M as its matrix. The point is to show that the existence of the complex derivative of f at z makes the defining condition (1) hold for F and L at (x, y).

First note that if we make $\mathbf{R}^2$ correspond to $\mathbf{C}$ by

$$\begin{bmatrix} x \\ y \end{bmatrix} \longleftrightarrow x + i\,y$$

(here x and y are general, not the particular x and y of the statement to be proved), then the correspondence extends to

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \longleftrightarrow (a + i\,b)(x + i\,y). \tag{2}$$

You should verify this to be sure you follow what's going on. Also, the correspondence preserves absolute value, i.e.,

$$\left| \begin{bmatrix} x \\ y \end{bmatrix} \right| = |x + i\,y|, \tag{3}$$

where the first absolute value is on $\mathbf{R}^2$ and the second one on $\mathbf{C}$.

We are assuming that the complex-valued function f of the complex variable z is complex-differentiable at the point $z = x + i\,y$, and we want to show that therefore its counterpart vector-valued function F of the vector variable (x, y) is vector-differentiable at (x, y), with the matrix of its derivative being the displayed M.

Let L be the linear map whose matrix is M, and let $\Delta z = h + i\,k$. Then as a special case of (2)

$$L(h,k) = M \begin{bmatrix} h \\ k \end{bmatrix} \longleftrightarrow (u_x(x,y) + iv_x(x,y))(h + i\,k) = f'(z)\Delta z.$$

Therefore by (3), the quotient of $\mathbf{R}^2$-absolute values in the defining property (1) of the vector derivative is also a quotient of $\mathbf{C}$-absolute values, and since $\mathbf{C}$ is a field this is in turn is the $\mathbf{C}$-absolute value of the quotient,

$$\begin{aligned} &\frac{|F(x+h, y+k) - F(x,y) - L(h,k)|}{|(h,k)|} \\ &= \frac{|f(z+\Delta z) - f(z) - f'(z)\Delta z|}{|\Delta z|} \\ &= \left| \frac{f(z+\Delta z) - f(z)}{\Delta z} - f'(z) \right| \end{aligned}$$

The third of the three equal quantities certainly goes to 0 as $\Delta z \to 0$ by the very definition of the complex derivative,

$$f'(z) = \lim_{z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}.$$

And $\Delta z \to 0$ in $\mathbf{C}$ if and only if $(h, k) \to (0, 0)$ in $\mathbf{R}^2$, so the first of the three equal quantities goes to 0 as $(h, k) \to (0, 0)$. This is the defining condition (1) for F and L. Therefore the linear map L whose matrix is M meets the condition to be $DF_{(x,y)}$, as desired.

In hindsight, this argument is inevitable if one remembers that the definition of the vector derivative was concocted to get around the general absence of division in the vector space $\mathbf{R}^n$. When $\mathbf{R}^2$ also has the field structure of $\mathbf{C}$, recovering division, the field definition of derivative naturally encompasses the vector definition. The fact that $\mathbf{C}$ has the algebraic structure of a field as well as the geometric structure of a plane is part of why the complex derivative has such rich properties.

Finally, now that we're warmed up, we may as well prove a converse statement as well by reversing the steps:

Suppose that the vector derivative $DF_{(x,y)}$ exists and that the Cauchy–Riemann equations are satisfied at (x, y),

$$u_x(x, y) = v_y(x, y), \qquad u_y(x, y) = -v_x(x, y).$$

Then the complex derivative $f'(z)$ also exists at the point $z = x + iy$.

Proof. Again let

$$M = \begin{bmatrix} u_x(x, y) & -v_x(x, y) \\ v_x(x, y) & u_x(x, y) \end{bmatrix},$$

and again let L be the corresponding linear map. The given condition that the vector derivative exists is

$$\lim_{(h,k)\to(0,0)} \frac{|F(x + h, y + k) - F(x, y) - L(h, k)|}{|(h, k)|} = 0.$$

This translates into the complex condition

$$\lim_{\Delta z\to 0} \frac{|f(z + \Delta z) - f(z) - (u_x(x, y) + iv_x(x, y))\Delta z|}{|\Delta z|} = 0,$$

or, since we may drop the absolute values signs and do algebra using the complex field structure,

$$\lim_{\Delta z\to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = u_x(x, y) + iv_x(x, y).$$

That is, the only possible candidate $u_x(x, y) + iv_x(x, y)$ for $f'(z)$ indeed works.