COMPLEX DIFFERENTIABILITY AND VECTOR DIFFERENTIABILITY

A complex-valued function on a complex region can be viewed as an $\mathbb{R}^2$-valued function on an $\mathbb{R}^2$-region, and conversely. Consider a region $\Omega_{\mathbb{C}}$ in $\mathbb{C}$, which can be identified with a region $\Omega_{\mathbb{R}^2}$ in $\mathbb{R}^2$, and consider a function $f : \Omega_{\mathbb{C}} \longrightarrow \mathbb{C}$, which can be identified with a function $F = (u, v) : \Omega_{\mathbb{R}^2} \longrightarrow \mathbb{R}^2$. Identify any point $z = x + iy$ in $\Omega_{\mathbb{C}}$ with the vector (x, y) in $\Omega_{\mathbb{R}^2}$. This writeup shows that for any such point, the complex condition

$$f \text{ is differentiable as a complex function at } z$$

is equivalent to the two vector conditions,

$$\left.\begin{cases} F = (u, v) \text{ is differentiable as a vector function} \\ (u_x, v_x) = (v_y, -u_y) \end{cases}\right\} \text{ at } (x, y).$$

Especially, if u_x, u_y, v_x, v_y are continuous on Ω, and $(u_x, v_x) = (v_y, -u_y)$ on $\Omega_{\mathbb{R}^2}$ then $f = u + iv$ is complex-differentiable on $\Omega_{\mathbb{C}}$.

1. The Little-oh Notation

Recall a quantified notion of smallness that figures in the characterizing property of the derivative.

Definition 1.1. *Let U be an open superset of $\mathbf{0}$ in $\mathbb{R}^n$. A function*

$$\varphi : U \longrightarrow \mathbb{R}^m$$

is called an $\mathrm{o}(h)$*-function if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that*

$$\left.\begin{cases} h \in U \\ |h| \le \delta \end{cases}\right\} \implies |\varphi(h)| \le \varepsilon |h|.$$

The geometric intuition is that the graph of φ passes horizontally through the origin.

A pair of conditions equivalent to the $o(h)$ condition is

$$\lim_{h \to \mathbf{0}} \frac{|\varphi(h)|}{|h|} = 0 \quad \text{and} \quad \varphi(\mathbf{0}) = \mathbf{0}.$$

However, the first condition requires vigilance to avoid dividing by 0 in the middle of some complicated situation (such errors creep into the texts, even good texts by serious, able authors), and the second condition is a nuisance to check every time.

2. Complex Differentiability

Let $\Omega \subset \mathbb{C}$ be a region, and let z be a point of Ω. Let $c \in \mathbb{C}$ be some fixed complex number. A succession of conditions are equivalent:

$$\begin{aligned}
\lim_{\ell \to 0_{\mathbb{C}}} \frac{f(z+\ell) - f(z)}{\ell} = c &\iff \lim_{\ell \to 0_{\mathbb{C}}} \frac{f(z+\ell) - f(z) - c\,\ell}{\ell} = 0_{\mathbb{C}} \\
&\iff \left\{ \begin{aligned} &\lim_{\ell \to 0_{\mathbb{C}}} \frac{|f(z+\ell) - f(z) - c\,\ell|}{|\ell|} = 0_{\mathbb{R}} \\ &f(z+\ell) - f(z) - c\,\ell \Big|_{\ell = 0_{\mathbb{C}}} = 0_{\mathbb{C}} \end{aligned} \right\} \\
&\iff f(z+\ell) - f(z) - c\,\ell = o(\ell).
\end{aligned}$$

Define f to be differentiable at z with derivative $f'(z) = c$ if the first and/or the last of the displayed conditions hold. These two equivalent definitions of the derivative—either as a limit of difference-quotients or as a multiplication-factor satisfying a characterizing property—are symbolically identical to their counterparts from the calculus of one real variable.

3. A Compatibility of Multiplications

Recall the set

$$S = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

from the first homework assignment. View elements of $\mathbb{R}^2$ as column vectors. The correspondences from complex numbers to certain matrices to column vectors,

$$x + iy \xrightarrow{m} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \xrightarrow{v} \begin{bmatrix} x \\ y \end{bmatrix},$$

naturally identify three normed vector spaces over $\mathbb{R}$,

$$(\mathbb{C}, |\ |_{\mathbb{C}}), \qquad (S, \sqrt{\det}\,), \qquad (\mathbb{R}^2, |\ |_{\mathbb{R}^2}).$$

While $\mathbb{C}$ and S are in fact algebras whose multiplications are compatible under their identification, the multiplication that takes the third space back to itself is matrix-by-vector multiplication, in particular

$$S \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

It is easy to check that the following diagram commutes:

$$\begin{tikzcd}
S \times S \arrow[r] \arrow[d] & S \times \mathbb{R}^2 \arrow[d] \\
S \arrow[r] & \mathbb{R}^2
\end{tikzcd}$$

Specifically, both ways around the square give

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \times \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \longmapsto \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix}.$$

That is, the left column-vector of the product of matrices is the left matrix times the left column-vector of the right matrix.

4. Complex Differentiability and Vector Differentiability

Let $\Omega \subset \mathbb{C}$ be a region, and consider a map

$$f : \Omega \longrightarrow \mathbb{C}, \qquad f(z) = w.$$

We may also view Ω as a subset of $\mathbb{R}^2$ and f as a vector-valued function

$$F : \Omega \to \mathbb{R}^2, \qquad F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} u \\ v \end{bmatrix}.$$

Let $c = a + ib$ be a fixed complex number, and let $\ell = h + ik$ be a variable complex number close to $0_{\mathbb{C}}$. The complex quantity

$$f(z+\ell) - f(z) - c\,\ell$$

is taken by the correspondence between $\mathbb{C}$ and S to the matrix

$$m(f(z+\ell)) - m(f(z)) - m(c)\,m(\ell),$$

which in turn is taken by the correspondence between S and $\mathbb{R}^2$ to the column vector (using the last formula from the previous section)

$$F(\begin{bmatrix} x+h \\ y+k \end{bmatrix}) - F(\begin{bmatrix} x \\ y \end{bmatrix}) - \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}.$$

Because the correspondences preserve norms, we have shown that two conditions are equivalent,

$$f(z+\ell) - f(z) - c\,\ell \quad \text{is} \quad o(\ell)$$

if and only if

$$F(\begin{bmatrix} x+h \\ y+k \end{bmatrix}) - F(\begin{bmatrix} x \\ y \end{bmatrix}) - \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \quad \text{is} \quad o(\begin{bmatrix} h \\ k \end{bmatrix}).$$

That is, by our definition of the complex derivative and by the standard definition of the vector derivative,

$$f \text{ is differentiable at } z = x + iy \text{ with derivative } f'(z) = c = a + ib$$

if and only if

$$F \text{ is differentiable at } \begin{bmatrix} x \\ y \end{bmatrix} \text{ with derivative matrix } F'(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

5. The Cauchy–Riemann Equations

Continue to denote our vector function

$$F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}.$$

The form of the derivative matrix in the previous display shows that if F describes a complex function f that is differentiable at a point $z = x + iy$ then the partial derivatives of the component functions of F satisfy the relations

$$u_x(x,y) = v_y(x,y) \quad \text{and} \quad v_x(x,y) = -u_y(x,y).$$

The relations are the *Cauchy–Riemann equations*.

The Cauchy–Riemann relations are more traditionally derived from the formula

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z},$$

by first letting $\Delta z = h$ so that the approach is horizontal and then letting $\Delta z = ik$ so that the approach is vertical. That is, we have both

$$\begin{aligned} f'(z) &= \lim_{h\to 0_{\mathbb{R}}} \frac{f(z+h)-f(z)}{h} \\ &= \lim_{h\to 0_{\mathbb{R}}} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\ &= u_x(x,y)+iv_x(x,y) \end{aligned}$$

and

$$\begin{aligned} f'(z) &= \lim_{k\to 0_{\mathbb{R}}} \frac{f(z+ik)-f(z)}{ik} \\ &= \lim_{k\to 0_{\mathbb{R}}} \frac{u(x,y+k)+iv(x,y+k)-u(x,y)-iv(x,y)}{ik} \\ &= v_y(x,y)-iu_y(x,y). \end{aligned}$$

Thus the complex differentiability of the function $f = u+iv$ at the point $z = x+iy$ implies the Cauchy–Riemann equations there. But the method of this writeup has shown more. At each point we have:

$$\textit{complex differentiability} \iff \left\{ \begin{array}{l} \textit{vector differentiability} \\ \quad \textit{and} \\ \textit{the Cauchy–Riemann equations} \end{array} \right\}.$$

Complex contour integration will show that complex differentiability over a region gives power series representation. Consequently, we have the astonishing result that vector differentiability and the Cauchy–Riemann equations over a region give real-analytic representation in turn.

For example, consider the complex exponential function,

$$f(z) = e^z, \quad z \in \mathbb{C}.$$

Because $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$, the corresponding vector map is

$$(u(x,y), v(x,y)) = (e^x \cos y, e^x \sin y).$$

It matrix of partial derivatives is

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

The partial derivatives are continuous, giving vector differentiability, and the Cauchy–Riemann equations hold. Thus the complex exponential function is differentiable everywhere, and its derivative at each input is the complex number described by the first column of the derivative matrix,

$$f'(z) = f'(x+iy) = e^x \cos y + ie^x \sin y = e^z.$$