SKETCH OF FUNCTION THEORY ON COMPLEX TORI

In class we have shown that if $\Lambda \subset \mathbb{C}$ is a lattice and

$$f : \mathbb{C}/\Lambda \rightarrow \hat{\mathbb{C}}$$

is a nonzero meromorphic function then three necessary conditions follow from short contour integral calculations:

1. $\sum_{c \in \mathbb{C}/\Lambda} \text{res}_c(f) = 0$. That is, the sum of the residues of $f$ is zero. This condition rules out the possibility of a meromorphic function on $\mathbb{C}/\Lambda$ having only one simple pole.

2. $\sum_{c \in \mathbb{C}/\Lambda} \text{ord}_c(f) = 0$. That is, the net order of vanishing of $f$ is zero. (Indeed, this holds with any compact Riemann surface in place of $\mathbb{C}/\Lambda$: triangulate the surface so that the triangle-sides avoid the finitely many zeros and poles of $f$; then the sum of the integrals of $f'(z)/f(z)$ around all the triangles is zero by cancellation, but also it is the net order of vanishing.)

3. $\sum_{c \in \mathbb{C}/\Lambda} \text{ord}_c(f) \cdot c = 0$ in $\mathbb{C}/\Lambda$. That is, the sum of the points where $f$ has zeros and poles, each such point $c$ summed as many times as $f$ vanishes there, is zero under the group law of $\mathbb{C}/\Lambda$.

These conditions are also sufficient. Specifically, after introducing some building-block functions in the next section, this writeup constructs a $\Lambda$-periodic function with any feasible prescribed vanishing behavior, and also this writeup constructs a $\Lambda$-periodic function with any feasible prescribed principal parts.

The second part of this writeup shows that the field of meromorphic functions on a complex torus is the field of rational functions in the Weierstrass $\wp$-function and its derivative.

1. Constructions

1.1. Weierstrass’s $\sigma$-function, $\zeta$-function, and $\wp$-function. The Weierstrass $\sigma$-function,

$$\sigma : \mathbb{C} \rightarrow \mathbb{C},$$

is

$$\sigma(z) = z \prod_{\omega \in \Lambda} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}.$$  

Since this function has simple zeros at the two-dimensional lattice $\Lambda \subset \mathbb{C}$ just as the function $s(x) = \sin \pi x$ has simple zeros at the one-dimensional lattice $\mathbb{Z} \subset \mathbb{R}$, it is named $\sigma$ by analogy.

(The exponential factors are needed to make the infinite product converge. A full explanation of this would take us too far afield, but the basic idea is that for the product to converge to a nonzero value, the value needs to be the exponential of the sum of the logarithms of the multiplicands. The relevant question becomes whether the sum

$$\log z + \sum_{\omega \in \Lambda} \log \left((1 - z/\omega)e^{z/\omega + \frac{1}{2}(z/\omega)^2}\right)$$

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converges absolutely and uniformly on compacta. And it does, because for small enough $|z|$ the logarithm of the product is the sum of the logarithms,

$$\log((1 - z/e^{z/\omega} + \frac{1}{2}(z/\omega)^2) = \log(1 - z/\omega) + z/\omega + \frac{1}{2}(z/\omega)^2,$$

and the power series expansion of the logarithm is

$$\log(1 - z/\omega) = -\frac{z}{\omega} - \frac{1}{2}\frac{(z/\omega)^2}{\omega} - \frac{1}{3}\frac{(z/\omega)^3}{\omega^2} - \cdots.$$ 

Thus the summand is $O((z/\omega)^3)$. As discussed in class, this is small enough to make the sum converge nicely. Consequently the infinite product $\sigma(z)$ converges to a holomorphic function on $\mathbb{C}$. The theory of infinite products is covered in many texts. See, for example, Complex Functions by Jones and Singerman.)

The Weierstrass $\zeta$-function,

$$\zeta: \mathbb{C} \rightarrow \hat{\mathbb{C}},$$

emphatically is not the Euler–Riemann $\zeta$-function, but instead is

$$\zeta(z) = \log(\sigma(z)) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega \in \Lambda}^\prime \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

This function has simple poles with residue 1 at the lattice points, analogously to the logarithmic derivative $\pi \cot \pi x$ of $\sin \pi x$, but it isn’t quite periodic with respect to $\Lambda$. However, let $\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}$ where $\omega_1/\omega_2 \in \mathcal{H}$. Since the Weierstrass $\wp$-function $\wp = -\zeta'$ is $\Lambda$-periodic, the quantities

$$\eta_j = \zeta(z + \omega_j) - \zeta(z), \quad j = 1, 2$$

are independent of $z$, i.e., they are lattice constants. Now we have the transformation laws

$$\zeta(z + \omega_j) = \zeta(z) + \eta_j, \quad j = 1, 2.$$

Consequently,

$$\left( \log \left( \frac{\sigma(z + \omega_j)}{\sigma(z)} \right) \right) = \zeta(z + \omega_j) - \zeta(z) = \eta_j, \quad j = 1, 2,$$

and thus for some constant $c$,

$$\log \left( \frac{\sigma(z + \omega_j)}{\sigma(z)} \right) = \eta_j z + c, \quad j = 1, 2,$$

or

$$\sigma(z + \omega_j) = \sigma(z) e^{\eta_j z} e^c, \quad j = 1, 2,$$

To determine $e^c$, note that the definition of $\sigma$ shows that $\sigma$ is odd. Therefore, setting $z = -\omega_j/2$ gives

$$\sigma(\omega_j/2) = -\sigma(\omega_j/2) e^{-\eta_j \sigma_j/2} e^c.$$

And so $e^c = -e^{\eta_j \sigma_j/2}$, giving

$$\sigma(z + \omega_j) = -\sigma(z) e^{\eta_j (z + \omega_j/2)}, \quad j = 1, 2.$$

Incidentally, the lattice constants satisfy the Legendre relation,

$$\eta_2 \omega_1 - \eta_1 \omega_2 = 2\pi i.$$
1.2. **Constructing a function with specified vanishing.** Now let $n$ be a positive integer, and consider a set of data $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$ where $a_1$ through $a_n$ can contain repeats, as can $b_1$ through $b_n$, but no $a_i$ and $b_j$ are equal modulo the lattice $\Lambda$. Suppose further that

$$\sum_i a_i - \sum_i b_i \in \Lambda.$$ 

This condition forces $n \geq 2$. We want to define a meromorphic function $f : \mathbb{C}/\Lambda \to \hat{\mathbb{C}}$ with zeros at the $a_i$, the degree of each zero being the number of times that the corresponding $a_i$ repeats, and similarly for poles at the $b_i$. Such a function will satisfy the second of our three necessary conditions, $\sum_{c} \text{ord}_c(f) = 0$, and also the third, $\sum_{c} \text{ord}_c(f) \cdot c = 0$ in $\mathbb{C}/\Lambda$.

Translating $b_n$ by some lattice element $\lambda \in \Lambda$, which has no effect on the coset $b_n + \Lambda \in \mathbb{C}/\Lambda$, we may assume that in fact $\sum_i a_i - \sum_i b_i = 0$.

Now consider the function $f : \mathbb{C} \to \hat{\mathbb{C}}$, $f(z) = \prod_i \sigma(z - a_i) / \prod_i \sigma(z - b_i)$.

This function is meromorphic, and it has the specified zeros and poles. The question is whether $f$ is $\Lambda$-periodic. So compute for $j = 1, 2$ that

$$f(z + \omega_j) = \prod_i \sigma(z - a_i + \omega_j) / \prod_i \sigma(z - b_i + \omega_j)$$

$$= (-1)^n \prod_i \sigma(z - a_i) e^{\eta_j(z - a_i + \omega_j/2)} / \prod_i \sigma(z - b_i) e^{\eta_j(z - b_i + \omega_j/2)}$$

$$= \prod_i \sigma(z - a_i) / \prod_i \sigma(z - b_i) \prod_i e^{\eta_j(b_i - a_i)}$$

$$= \prod_i \sigma(z - a_i) / \prod_i \sigma(z - b_i) e^{\eta_j \sum_i (b_i - a_i)}$$

$$= \prod_i \sigma(z - a_i) / \prod_i \sigma(z - b_i)$$

$$= f(z).$$

Thus $f$ is indeed $\Lambda$-periodic, giving a meromorphic function on the torus with the specified zeros and poles.

1.3. **Constructing a function with specified principal parts.** Recall that the **Weierstrass $\wp$-function**, $\wp : \mathbb{C} \to \hat{\mathbb{C}}$

is

$$\wp(z) = -\zeta'(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda}^\prime \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$
Define also for each integer \( k \geq 3 \),
\[
F_k : \mathbb{C} \rightarrow \mathbb{C}, \quad F_k(z) = \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^k}.
\]
(Thus \( F_k = (-1)^k \psi^{(k-2)}/(k-1)! \).) Recall that the Weierstrass \( \zeta \)-function has simple poles with residue 1 at the lattice points \( \omega \in \Lambda \). More specifically, its Laurent series at 0 is
\[
\zeta(z) = \frac{1}{z} + \text{holomorphic in } z.
\]
Similarly, the Weierstrass \( \wp \)-function Laurent series has a double pole at 0 and Laurent series
\[
\wp(z) = \frac{1}{z^2} + \text{holomorphic in } z,
\]
while for \( k \geq 3 \) the functions \( F_k \) have \( k \)-fold poles at 0 and Laurent series
\[
F_k(z) = \frac{1}{z^k} + \text{holomorphic in } z.
\]
Now let \( z_1 \) through \( z_m \) be distinct modulo \( \Lambda \), and consider a set of principal part data
\[
\begin{align*}
P_1(z) &= \frac{c_{1,1}}{z - z_1} + \frac{c_{1,2}}{(z - z_1)^2} + \cdots + \frac{c_{1,n_1}}{(z - z_1)^{n_1}} \\
P_2(z) &= \frac{c_{2,1}}{z - z_2} + \frac{c_{2,2}}{(z - z_2)^2} + \cdots + \frac{c_{2,n_2}}{(z - z_2)^{n_2}} \\
&\quad\vdots \\
P_m(z) &= \frac{c_{m,1}}{z - z_m} + \frac{c_{m,2}}{(z - z_m)^2} + \cdots + \frac{c_{m,n_m}}{(z - z_m)^{n_m}}
\end{align*}
\]
where the coefficients of the minus-first powers sum to zero,
\[
c_{1,1} + \cdots + c_{m,1} = 0.
\]
These data might describe the principal parts of a meromorphic function on \( \mathbb{C}/\Lambda \) at its poles, since the residues of the putative function sum to zero.

The meromorphic function on \( \mathbb{C} \) with the desired principal parts is
\[
f(z) = c_{1,1}\zeta(z - z_1) + c_{1,2}\wp(z - z_1) + \cdots + c_{1,n_1}F_{n_1}(z - z_1) \\
+ c_{2,1}\zeta(z - z_2) + c_{2,2}\wp(z - z_2) + \cdots + c_{2,n_2}F_{n_2}(z - z_2) \\
+ \cdots \\
+ c_{m,1}\zeta(z - z_m) + c_{m,2}\wp(z - z_m) + \cdots + c_{m,n_m}F_{n_m}(z - z_m).
\]
More briefly, \( f(z) = \sum_{i,j} c_{i,j}F_{j}(z - z_j) \) where now \( F_1 = \zeta \) and \( F_2 = \wp \). The question is whether \( f \) is \( \Lambda \)-periodic. Since the Weierstrass \( \wp \)-function and its derivatives are \( \Lambda \)-periodic, the question bears only on the subfunction
\[
g(z) = c_{1,1}\zeta(z - z_1) + \cdots + c_{m,1}\zeta(z - z_m) = \sum_{i=1}^m c_{i,1}\zeta(z - z_i).
\]
Compute for \( j = 1, 2 \) that
\[
g(z + \omega_j) = \sum_{i=1}^m c_{i,1}\zeta(z - z_i + \omega_j) = \sum_{i=1}^m c_{i,1}(\zeta(z - z_i) + \eta_j) = g(z) + \eta_j \sum_{i=1}^m c_{i,1}.
\]
And thus \( g(z + \omega_j) = g(z) \) because \( \sum_i c_{i,1} = 0 \).
2. THE FIELD OF MEROMORPHIC FUNCTIONS ON A COMPLEX TORUS

Let \( \Lambda \) be a lattice, and let \( \wp \) be its associated Weierstrass function. We show that the field of meromorphic functions on \( \mathbb{C}/\Lambda \)—or, equivalently, the field of \( \Lambda \)-periodic meromorphic functions on \( \mathbb{C} \)—is as simple as it possibly could be: it is only the field of rational functions in \( \wp \) and \( \wp' \),

\[ \mathbb{C}(\wp, \wp'), \]

and in fact this field is

\[ \mathbb{C}(\wp)[\wp'] = \{ f(\wp) + \wp' g(\wp) : f, g \text{ rational functions} \}. \]

So up to isomorphism, the function field is generated by two transcendental quantities over \( \mathbb{C} \) that satisfy an algebraic relation,

\[ \mathbb{C}(x, y)/(y^2 = 4x^3 - g_2x - g_3). \]

To establish the desired result, consider any meromorphic function \( f \) on \( \mathbb{C}/\Lambda \), and introducing two resulting even functions,

\[ f_1(z) = \frac{f(z) + f(-z)}{2}, \quad f_2(z) = \frac{f(z) - f(-z)}{2\wp'(z)}. \]

Then we have the decomposition

\[ f(z) = f_1(z) + \wp'(z)f_2(z). \]

This reduces the problem to showing that the field of even meromorphic functions on \( \mathbb{C}/\Lambda \) is \( \mathbb{C}(\wp) \).

So now consider any even meromorphic function \( f \) on \( \mathbb{C}/\Lambda \), where \( \Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z} \).

Its expansion about 0 is

\[ f(z) = \sum_{n \geq \nu_0(f)} a_n z^n, \quad z \text{ near } 0, \]

and so all powers of \( z \) in this expansion are even. In particular, the vanishing order \( \nu_0(f) \) is even. Similarly, the expansion of \( f \) about \( \omega_1/2 \) is

\[ f(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n (z - \frac{\omega_1}{2})^n, \quad z \text{ near } 0. \]

Define a related meromorphic function on \( \mathbb{C}/\Lambda \),

\[ g(z) = f(z + \frac{\omega_1}{2}). \]

To see that \( g \) is even because \( f \) is even and because \( \frac{\omega_1}{2} \) is its own inverse in \( \mathbb{C}/\Lambda \), compute

\[ g(-z) = f(-z + \frac{\omega_1}{2}) = f(-z - \frac{\omega_1}{2} + \omega_1) = f(-z - \frac{\omega_1}{2}) = f(z + \frac{\omega_1}{2}) = g(z). \]

Thus the order of \( g \) at 0 is even, as shown earlier in this paragraph. But the Laurent expansion of \( g \) about 0 is

\[ g(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n z^n, \quad z \text{ near } 0. \]

Thus \( \nu_{\omega_1/2}(f) \) is even. Similarly, \( \nu_{\omega_2/2}(f) \) and \( \nu_{(\omega_1+\omega_2)/2}(f) \) are even.
All points of $\mathbb{C}/\Lambda$ come in opposite pairs $\{\pm p\}$, other than (the cosets of) the four points $q = 0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$. Given the even meromorphic function $f$ on $\mathbb{C}/\Lambda$, consider the related function

$$\varphi(z) = \prod_p (\varphi(z) - \varphi(p))^{\nu_p(f)} \prod_q (\varphi(z) - \varphi(q))^{\nu_q(f)/2}.$$ 

The first product in the previous display chooses either point of each pair $\{\pm p\}$. The function $\varphi$ is a rational function in $\varphi$. Because $\varphi$ takes the values $q$ to order 2, the function $\varphi$ has the same order of vanishing as $f$ everywhere. Thus their quotient is analytic and doubly periodic, making it constant, and so $f$ is a rational function in $\varphi$ as well. This completes the argument.