SKETCH OF FUNCTION THEORY ON COMPLEX TORI

In class we have shown that if \( \Lambda \subset \mathbb{C} \) is a lattice and 
\[
f : \mathbb{C}/\Lambda \to \hat{\mathbb{C}}
\]
is a nonzero meromorphic function then three necessary conditions follow from short contour integral calculations:

1. \( \sum_{z \in \mathbb{C}/\Lambda} \text{ord}_z(f) = 0 \). That is, the net order of vanishing of \( f \) is zero. (Indeed, this holds with any compact Riemann surface in place of \( \mathbb{C}/\Lambda \): triangulate the surface so that the triangle-sides avoid the finitely many zeros and poles of \( f \); then the sum of the integrals of \( f'(z)/f(z) \) around all the triangles is zero by cancellation, but also it is the net order of vanishing.)

2. \( \sum_{z \in \mathbb{C}/\Lambda} \text{ord}_z(f) \cdot z = 0 \) in \( \mathbb{C}/\Lambda \). That is, the sum of the points where \( f \) has zeros and poles, each such point \( z \) summed as many times as \( f \) vanishes there, is zero under the group law of \( \mathbb{C}/\Lambda \).

3. \( \sum_{z \in \mathbb{C}/\Lambda} \text{res}_z(f) = 0 \). That is, the sum of the residues of \( f \) is zero. This condition rules out the possibility of a meromorphic function on \( \mathbb{C}/\Lambda \) having only one simple pole.

These conditions are also sufficient. Specifically, after introducing some building-block functions in the next section, this writeup constructs a \( \Lambda \)-periodic function with any feasible prescribed vanishing behavior, and also this writeup constructs a \( \Lambda \)-periodic function with any feasible prescribed principal parts.

1. Weierstrass’s \( \sigma \)-function, \( \zeta \)-function, and \( \wp \)-function

The Weierstrass \( \sigma \)-function,
\[
\sigma : \mathbb{C} \to \mathbb{C},
\]
is
\[
\sigma(z) = z \prod_{\omega \in \Lambda}' \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}.
\]
Since this function has simple zeros at the two-dimensional lattice \( \Lambda \subset \mathbb{C} \) just as the function \( s(x) = \sin \pi x \) has simple zeros at the one-dimensional lattice \( \mathbb{Z} \subset \mathbb{R} \), it is named \( \sigma \) by analogy.

(The exponential factors are needed to make the infinite product converge. A full explanation of this would take us too far afield, but the basic idea is that for the product to converge to a nonzero value, the value needs to be the exponential of the sum of the logarithms of the multiplicands. The relevant question becomes whether the sum
\[
\log z + \sum_{\omega \in \Lambda}' \log \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}
\]
converges absolutely and uniformly on compacta. And it does, because for small enough \(|z|\) the logarithm of the product is the sum of the logarithms,
\[
\log \left((1 - z/\omega) e^{z/\omega + \frac{1}{2}(z/\omega)^2}\right) = \log(1 - z/\omega) + z/\omega + \frac{1}{2}(z/\omega)^2,
\]
and the power series expansion of the logarithm is
\[
\log(1 - z/w) = -z/w - \frac{1}{2}(z/w)^2 - \frac{1}{3}(z/w)^3 - \cdots.
\]
Thus the summand is \(O((z/w)^3)\). As discussed in class, this is small enough to make the sum converge nicely. Consequently the infinite product \(\sigma(z)\) converges to a holomorphic function on \(\mathbb{C}\). The theory of infinite products is covered in many texts. See, for example, \textit{Complex Functions} by Jones and Singerman.)

The \textit{Weierstrass} \(\zeta\)-function,

\[
\zeta : \mathbb{C} \longrightarrow \hat{\mathbb{C}},
\]

emphatically is \textit{not} the Euler–Riemann \(\zeta\)-function, but instead is

\[
\zeta(z) = \log(\sigma(z))' = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega \in \Lambda} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).
\]

This function has simple poles with residue 1 at the lattice points, analogously to the logarithmic derivative \(\pi \cot \pi x\) of \(\sin \pi x\), but it isn’t quite periodic with respect to \(\Lambda\). However, let \(\Lambda = \omega_1 \mathbb{Z} \oplus \omega_2 \mathbb{Z}\) where \(\omega_1/\omega_2 \in \mathcal{H}\). Since the Weierstrass \(\wp\)-function \(\wp = -\zeta'\) is \(\Lambda\)-periodic, the quantities

\[
\eta_j = \zeta(z + \omega_j) - \zeta(z), \quad j = 1, 2
\]

are independent of \(z\), i.e., they are lattice constants. Now we have the transformation laws

\[
\zeta(z + \omega_j) = \zeta(z) + \eta_j, \quad j = 1, 2
\]

Consequently,

\[
\log \left( \frac{\sigma(z + \omega_j)}{\sigma(z)} \right)' = \zeta(z + \omega_j) - \zeta(z) = \eta_j, \quad j = 1, 2,
\]

and thus for some constant \(c\),

\[
\log \left( \frac{\sigma(z + \omega_j)}{\sigma(z)} \right) = \eta_j z + c, \quad j = 1, 2.
\]

To determine \(c\), note that the definition of \(\sigma\) shows that \(\sigma\) is odd. Therefore, setting \(z = -\omega_j/2\) gives

\[
\frac{i\pi}{2} = \log \left( \frac{\sigma(\omega_j/2)}{\sigma(-\omega_j/2)} \right) = -\eta_j \omega_j/2 + c.
\]

And so

\[
\sigma(z + \omega_j) = -\sigma(z)e^{\eta_j(z+\omega_j/2)}, \quad j = 1, 2
\]

Incidentally, the lattice constants satisfy the \textit{Legendre relation},

\[
\eta_2 \omega_1 - \eta_1 \omega_2 = 2\pi i.
\]
2. Constructing a Function Having Specified Vanishing

Now let \( n \) be a positive integer, and consider a set of data

\[
an_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}
\]

where \( a_1 \) through \( a_n \) can contain repeats, as can \( b_1 \) through \( b_n \), but no \( a_i \) and \( b_j \) are equal modulo the lattice \( \Lambda \). Suppose further that

\[
\sum_i a_i - \sum_i b_i \in \Lambda.
\]

We want to define a meromorphic function

\[
f : \mathbb{C}/\Lambda \longrightarrow \hat{\mathbb{C}}
\]

with zeros at the \( a_i \), the degree of each zero being the number of times that the corresponding \( a_i \) repeats, and similarly for poles at the \( b_i \). Such a function will satisfy the first of our three necessary conditions, \( \sum \text{ord}_z(f) = 0 \), and also the second, \( \sum \text{ord}_z(f) \cdot z = 0 \) in \( \mathbb{C}/\Lambda \).

Translating \( b_n \) by some lattice element \( \lambda \in \Lambda \), which has no effect on the coset \( b_n + \Lambda \in \mathbb{C}/\Lambda \), we may assume that in fact

\[
\sum_i a_i - \sum_i b_i = 0.
\]

Now consider the function

\[
f : \mathbb{C} \longrightarrow \hat{\mathbb{C}}, \quad f(z) = \prod_i \frac{\sigma(z - a_i)}{\sigma(z - b_i)}.
\]

This function is meromorphic, and it has the specified zeros and poles. The question is whether it is \( \Lambda \)-periodic. So compute for \( j = 1, 2 \) that

\[
f(z + \omega_j) = \prod_i \frac{\sigma(z - a_i + \omega_j)}{\sigma(z - b_i + \omega_j)}
\]

\[
= (-1)^n \prod_i \sigma(z - a_i) e^{\eta_j (z - a_i + \omega_j/2)}
\]

\[
= (-1)^n \prod_i \sigma(z - b_i) e^{\eta_j (z - b_i + \omega_j/2)}
\]

\[
= \prod_i \sigma(z - a_i) \prod_i \sigma(z - b_i) e^{\eta_j (b_i - a_i)}
\]

\[
= \prod_i \sigma(z - a_i) e^{\eta_j \sum (b_i - a_i)}
\]

\[
= \prod_i \sigma(z - a_i) e^{\eta_j \sum (b_i - a_i)}
\]

\[
= \prod_i \sigma(z - a_i)
\]

Thus \( f \) is indeed \( \Lambda \)-periodic, giving a meromorphic function on the torus with the specified zeros and poles.

3. Constructing a Function Having Specified Principal Parts

Recall that the Weierstrass \( \wp \)-function,

\[
\wp : \mathbb{C} \longrightarrow \hat{\mathbb{C}}
\]
is
\[ \wp(z) = -\zeta'(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda} \left( \frac{1}{(z - \omega)^2} + \frac{1}{\omega^2} \right). \]

Define also for each integer \( k \geq 3 \),
\[ F_k : \mathbb{C} \rightarrow \hat{\mathbb{C}}, \quad F_k(z) = \sum_{\omega \in \Lambda} \frac{1}{(z - \omega)^k}. \]
(Thus \( F_k = (-1)^k \wp^{(k-2)}/(k-1)! \).) Recall that the Weierstrass \( \zeta \)-function has simple poles with residue 1 at the lattice points \( \omega \in \Lambda \). More specifically, its Laurent series at 0 is
\[ \zeta(z) = \frac{1}{z} + \text{holomorphic in } z. \]
Similarly, the Weierstrass \( \wp \)-function Laurent series has a double pole at 0 and Laurent series
\[ \wp(z) = \frac{1}{z^2} + \text{holomorphic in } z, \]
while for \( k \geq 3 \) the functions \( F_k \) have \( k \)-fold poles at 0 and Laurent series
\[ F_k(z) = \frac{1}{z^k} + \text{holomorphic in } z. \]

Now let \( z_1 \) through \( z_m \) be distinct modulo \( \Lambda \), and consider a set of principal part data,
\[ P_1(z) = \frac{c_{1,1}}{z - z_1} + \frac{c_{1,2}}{(z - z_1)^2} + \cdots + \frac{c_{1,n_1}}{(z - z_1)^{n_1}}, \]
\[ P_2(z) = \frac{c_{2,1}}{z - z_2} + \frac{c_{2,2}}{(z - z_2)^2} + \cdots + \frac{c_{2,n_2}}{(z - z_2)^{n_2}}, \]
\[ \vdots \]
\[ P_m(z) = \frac{c_{m,1}}{z - z_m} + \frac{c_{m,2}}{(z - z_m)^2} + \cdots + \frac{c_{m,n_m}}{(z - z_m)^{n_m}}, \]
where the coefficients of the negative-first powers sum to zero,
\[ c_{1,1} + \cdots + c_{m,1} = 0. \]
These data might describe the principal parts of a meromorphic function on \( \mathbb{C}/\Lambda \) at its poles, since the residues of the putative function sum to zero.

The meromorphic function on \( \mathbb{C} \) with the desired principal parts is
\[ f(z) = c_{1,1}\zeta(z - z_1) + c_{1,2}\wp(z - z_1) + \cdots + c_{1,n_1}F_{n_1}(z - z_1) \]
\[ + c_{2,1}\zeta(z - z_2) + c_{2,2}\wp(z - z_2) + \cdots + c_{2,n_2}F_{n_2}(z - z_2) \]
\[ \vdots \]
\[ + c_{m,1}\zeta(z - z_m) + c_{m,2}\wp(z - z_m) + \cdots + c_{m,n_m}F_{n_m}(z - z_m). \]
The question is whether \( f \) is \( \Lambda \)-periodic. Since the Weierstrass \( \wp \)-function and its derivatives are \( \Lambda \)-periodic, the question is bears only on the subfunction
\[ g(z) = c_{1,1}\zeta(z - z_1) + \cdots + c_{m,1}\zeta(z - z_m) = \sum_{i=1}^{m} c_{i,1}\zeta(z - z_i). \]
Compute for $j = 1, 2$ that
\[ g(z + \omega_j) = \sum_{i=1}^{m} c_{i,1} \zeta(z - z_i + \omega_j) = \sum_{i=1}^{m} c_{i,1} (\zeta(z - z_i) + \eta_j) = g(z) + \eta_j \sum_{i=1}^{m} c_{i,1}. \]
And thus $g(z + \omega_j) = g(z)$ because $\sum_{i} c_{i,1} = 0$. 