

SKETCH OF FUNCTION THEORY ON COMPLEX TORI

In class we have shown that if $\Lambda \subset \mathbf{C}$ is a lattice and

$$f : \mathbf{C}/\Lambda \longrightarrow \widehat{\mathbf{C}}$$

is a nonzero meromorphic function then three necessary conditions follow from short contour integral calculations:

- (1) $\sum_{z \in \mathbf{C}/\Lambda} \text{ord}_z(f) = 0$. That is, the net order of vanishing of f is zero. (Indeed, this holds with any compact Riemann surface in place of \mathbf{C}/Λ : triangulate the surface so that the triangle-sides avoid the finitely many zeros and poles of f ; then the sum of the integrals of $f'(z)/f(z)$ around all the triangles is zero by cancellation, but also it is the net order of vanishing.)
- (2) $\sum_{z \in \mathbf{C}/\Lambda} \text{ord}_z(f) \cdot z = 0$ in \mathbf{C}/Λ . That is, the sum of the points where f has zeros and poles, each such point z summed as many times as f vanishes there, is zero under the group law of \mathbf{C}/Λ .
- (3) $\sum_{z \in \mathbf{C}/\Lambda} \text{res}_z(f) = 0$. That is, the sum of the residues of f is zero. This condition rules out the possibility of a meromorphic function on \mathbf{C}/Λ having only one simple pole.

These conditions are also sufficient. Specifically, after introducing some building-block functions in the next section, this writeup constructs a Λ -periodic function with any feasible prescribed vanishing behavior, and also this writeup constructs a Λ -periodic function with any feasible prescribed principal parts.

1. WEIERSTRASS'S σ -FUNCTION, ζ -FUNCTION, AND \wp -FUNCTION

The *Weierstrass σ -function*,

$$\sigma : \mathbf{C} \longrightarrow \mathbf{C},$$

is

$$\sigma(z) = z \prod'_{\omega \in \Lambda} \left(1 - \frac{z}{\omega}\right) e^{z/\omega + \frac{1}{2}(z/\omega)^2}.$$

Since this function has simple zeros at the two-dimensional lattice $\Lambda \subset \mathbf{C}$ just as the function $s(x) = \sin \pi x$ has simple zeros at the one-dimensional lattice $\mathbf{Z} \subset \mathbf{R}$, it is named σ by analogy.

(The exponential factors are needed to make the infinite product converge. A full explanation of this would take us too far afield, but the basic idea is that for the product to converge to a nonzero value, the value needs to be the exponential of the sum of the logarithms of the multiplicands. The relevant question becomes whether the sum

$$\log z + \sum'_{\omega \in \Lambda} \log \left((1 - z/\omega) e^{z/\omega + \frac{1}{2}(z/\omega)^2} \right)$$

converges absolutely and uniformly on compacta. And it does, because for small enough $|z|$ the logarithm of the product is the sum of the logarithms,

$$\log \left((1 - z/\omega) e^{z/\omega + \frac{1}{2}(z/\omega)^2} \right) = \log(1 - z/\omega) + z/\omega + \frac{1}{2}(z/\omega)^2,$$

and the power series expansion of the logarithm is

$$\log(1 - z/w) = -z/w - \frac{1}{2}(z/w)^2 - \frac{1}{3}(z/w)^3 - \dots$$

Thus the summand is $\mathcal{O}((z/w)^3)$. As discussed in class, this is small enough to make the sum converge nicely. Consequently the infinite product $\sigma(z)$ converges to a holomorphic function on \mathbf{C} . The theory of infinite products is covered in many texts. See, for example, *Complex Functions* by Jones and Singerman.)

The *Weierstrass ζ -function*,

$$\zeta : \mathbf{C} \longrightarrow \widehat{\mathbf{C}},$$

emphatically is *not* the Euler–Riemann ζ -function, but instead is

$$\zeta(z) = \log(\sigma(z))' = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum'_{\omega \in \Lambda} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

This function has simple poles with residue 1 at the lattice points, analogously to the logarithmic derivative $\pi \cot \pi x$ of $\sin \pi x$, but it isn't quite periodic with respect to Λ . However, let $\Lambda = \omega_1 \mathbf{Z} \oplus \omega_2 \mathbf{Z}$ where $\omega_1/\omega_2 \in \mathcal{H}$. Since the Weierstrass \wp -function $\wp = -\zeta'$ is Λ -periodic, the quantities

$$\eta_j = \zeta(z + \omega_j) - \zeta(z), \quad j = 1, 2$$

are independent of z , i.e., they are lattice constants. Now we have the transformation laws

$$\boxed{\zeta(z + \omega_j) = \zeta(z) + \eta_j, \quad j = 1, 2.}$$

Consequently,

$$\log \left(\frac{\sigma(z + \omega_j)}{\sigma(z)} \right)' = \zeta(z + \omega_j) - \zeta(z) = \eta_j, \quad j = 1, 2,$$

and thus for some constant c ,

$$\log \left(\frac{\sigma(z + \omega_j)}{\sigma(z)} \right) = \eta_j z + c, \quad j = 1, 2.$$

To determine c , note that the definition of σ shows that σ is odd. Therefore, setting $z = -\omega_j/2$ gives

$$i\pi = \log \left(\frac{\sigma(\omega_j/2)}{\sigma(-\omega_j/2)} \right) = -\eta_j \omega_j/2 + c.$$

And so

$$\boxed{\sigma(z + \omega_j) = -\sigma(z) e^{\eta_j(z + \omega_j/2)}, \quad j = 1, 2.}$$

Incidentally, the lattice constants satisfy the *Legendre relation*,

$$\eta_2 \omega_1 - \eta_1 \omega_2 = 2\pi i.$$

2. CONSTRUCTING A FUNCTION HAVING SPECIFIED VANISHING

Now let n be a positive integer, and consider a set of data

$$a_1, \dots, a_n, b_1, \dots, b_n \in \mathbf{C}$$

where a_1 through a_n can contain repeats, as can b_1 through b_n , but no a_i and b_j are equal modulo the lattice Λ . Suppose further that

$$\sum_i a_i - \sum_i b_i \in \Lambda.$$

We want to define a meromorphic function

$$f : \mathbf{C}/\Lambda \longrightarrow \widehat{\mathbf{C}}$$

with zeros at the a_i , the degree of each zero being the number of times that the corresponding a_i repeats, and similarly for poles at the b_i . Such a function will satisfy the first of our three necessary conditions, $\sum \text{ord}_z(f) = 0$, and also the second, $\sum \text{ord}_z(f) \cdot z = 0$ in \mathbf{C}/Λ .

Translating b_n by some lattice element $\lambda \in \Lambda$, which has no effect on the coset $b_n + \Lambda \in \mathbf{C}/\Lambda$, we may assume that in fact

$$\sum_i a_i - \sum_i b_i = 0.$$

Now consider the function

$$f : \mathbf{C} \longrightarrow \widehat{\mathbf{C}}, \quad f(z) = \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)}.$$

This function is meromorphic, and it has the specified zeros and poles. The question is whether it is Λ -periodic. So compute for $j = 1, 2$ that

$$\begin{aligned} f(z + \omega_j) &= \frac{\prod_i \sigma(z - a_i + \omega_j)}{\prod_i \sigma(z - b_i + \omega_j)} \\ &= \frac{(-1)^n \prod_i \sigma(z - a_i) e^{\eta_j(z - a_i + \omega_j/2)}}{(-1)^n \prod_i \sigma(z - b_i) e^{\eta_j(z - b_i + \omega_j/2)}} \\ &= \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)} \prod_i e^{\eta_j(b_i - a_i)} \\ &= \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)} e^{\eta_j \sum_i (b_i - a_i)} \\ &= \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)} \\ &= f(z). \end{aligned}$$

Thus f is indeed Λ -periodic, giving a meromorphic function on the torus with the specified zeros and poles.

3. CONSTRUCTING A FUNCTION HAVING SPECIFIED PRINCIPAL PARTS

Recall that the *Weierstrass \wp -function*,

$$\wp : \mathbf{C} \longrightarrow \widehat{\mathbf{C}}$$

is

$$\wp(z) = -\zeta'(z) = \frac{1}{z^2} + \sum'_{\omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2} \right).$$

Define also for each integer $k \geq 3$,

$$F_k : \mathbf{C} \longrightarrow \widehat{\mathbf{C}}, \quad F_k(z) = \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^k}.$$

(Thus $F_k = (-1)^k \wp^{(k-2)}/(k-1)!$.) Recall that the Weierstrass ζ -function has simple poles with residue 1 at the lattice points $\omega \in \Lambda$. More specifically, its Laurent series at 0 is

$$\zeta(z) = \frac{1}{z} + \text{holomorphic in } z.$$

Similarly, the Weierstrass \wp -function Laurent series has a double pole at 0 and Laurent series

$$\wp(z) = \frac{1}{z^2} + \text{holomorphic in } z,$$

while for $k \geq 3$ the functions F_k have k -fold poles at 0 and Laurent series

$$F_k(z) = \frac{1}{z^k} + \text{holomorphic in } z.$$

Now let z_1 through z_m be distinct modulo Λ , and consider a set of principal part data,

$$\begin{aligned} P_1(z) &= \frac{c_{1,1}}{z-z_1} + \frac{c_{1,2}}{(z-z_1)^2} + \cdots + \frac{c_{1,n_1}}{(z-z_1)^{n_1}}, \\ P_2(z) &= \frac{c_{2,1}}{z-z_2} + \frac{c_{2,2}}{(z-z_2)^2} + \cdots + \frac{c_{2,n_2}}{(z-z_2)^{n_2}}, \\ &\vdots \\ P_m(z) &= \frac{c_{m,1}}{z-z_m} + \frac{c_{m,2}}{(z-z_m)^2} + \cdots + \frac{c_{m,n_m}}{(z-z_m)^{n_m}}. \end{aligned}$$

where the coefficients of the negative-first powers sum to zero,

$$c_{1,1} + \cdots + c_{m,1} = 0.$$

These data might describe the principal parts of a meromorphic function on \mathbf{C}/Λ at its poles, since the residues of the putative function sum to zero.

The meromorphic function on \mathbf{C} with the desired principal parts is

$$\begin{aligned} f(z) &= c_{1,1}\zeta(z-z_1) + c_{1,2}\wp(z-z_1) + \cdots + c_{1,n_1}F_{n_1}(z-z_1) \\ &\quad + c_{2,1}\zeta(z-z_2) + c_{2,2}\wp(z-z_2) + \cdots + c_{2,n_2}F_{n_2}(z-z_2) \\ &\quad \vdots \\ &\quad + c_{m,1}\zeta(z-z_m) + c_{m,2}\wp(z-z_m) + \cdots + c_{m,n_m}F_{n_m}(z-z_m). \end{aligned}$$

The question is whether f is Λ -periodic. Since the Weierstrass \wp -function and its derivatives are Λ -periodic, the question is bears only on the subfunction

$$g(z) = c_{1,1}\zeta(z-z_1) + \cdots + c_{m,1}\zeta(z-z_m) = \sum_{i=1}^m c_{i,1}\zeta(z-z_i).$$

Compute for $j = 1, 2$ that

$$g(z + \omega_j) = \sum_{i=1}^m c_{i,1} \zeta(z - z_i + \omega_j) = \sum_{i=1}^m c_{i,1} (\zeta(z - z_i) + \eta_j) = g(z) + \eta_j \sum_{i=1}^m c_{i,1}.$$

And thus $g(z + \omega_j) = g(z)$ because $\sum_i c_{i,1} = 0$.