SKETCH OF FUNCTION THEORY ON COMPLEX TORI

In class we have shown that if $\Lambda \subset \mathbb{C}$ is a lattice and

$$f : \mathbb{C}/\Lambda \longrightarrow \widehat{\mathbb{C}}$$

is a nonzero meromorphic function then three necessary conditions follow from short contour integral calculations:

(1) $\sum_{c\in\mathbb{C}/\Lambda} \operatorname{res}_c(f) = 0$. That is, the sum of the residues of f is zero. This condition rules out the possibility of a meromorphic function on $\mathbb{C}/\Lambda$ having only one simple pole.

(2) $\sum_{c\in\mathbb{C}/\Lambda} \operatorname{ord}_c(f) = 0$. That is, the net order of vanishing of f is zero. (Indeed, this holds with any compact Riemann surface in place of $\mathbb{C}/\Lambda$: triangulate the surface so that the triangle-sides avoid the finitely many zeros and poles of f; then the sum of the integrals of $f'(z)/f(z)$ around all the triangles is zero by cancellation, but also it is the net order of vanishing.)

(3) $\sum_{c\in\mathbb{C}/\Lambda} \operatorname{ord}_c(f)\cdot c = 0$ in $\mathbb{C}/\Lambda$. That is, the sum of the points where f has zeros and poles, each such point c summed as many times as f vanishes there, is zero under the group law of $\mathbb{C}/\Lambda$.

These conditions are also sufficient. Specifically, after introducing some building-block functions in the next section, this writeup constructs a Λ-periodic function with any feasible prescribed vanishing behavior, and also this writeup constructs a Λ-periodic function with any feasible prescribed principal parts.

The second part of this writeup shows that the field of meromorphic functions on a complex torus is the field of rational functions in the Weierstrass $\wp$-function and its derivative.

1. Constructions

1.1. **Weierstrass's σ-function, ζ-function, and $\wp$-function.**

The *Weierstrass σ-function*,

$$\sigma : \mathbb{C} \longrightarrow \mathbb{C},$$

is

$$\sigma(z) = z\prod_{\omega\in\Lambda}{}' \left(1-\frac{z}{\omega}\right) e^{z/\omega+\frac{1}{2}(z/\omega)^2}.$$

Since this function has simple zeros at the two-dimensional lattice $\Lambda \subset \mathbb{C}$ just as the function $s(x) = \sin \pi x$ has simple zeros at the one-dimensional lattice $\mathbb{Z} \subset \mathbb{R}$, it is named σ by analogy.

(The exponential factors are needed to make the infinite product converge. A full explanation of this would take us too far afield, but the basic idea is that for the product to converge to a nonzero value, the value needs to be the exponential of the sum of the logarithms of the multiplicands. The relevant question becomes whether the sum

$$\log z + \sum_{\omega\in\Lambda}{}' \log\left((1-z/\omega)e^{z/\omega+\frac{1}{2}(z/\omega)^2}\right)$$

converges absolutely and uniformly on compacta. And it does, because for small enough $|z|$ the logarithm of the product is the sum of the logarithms,

$$\log\left((1-z/\omega)e^{z/\omega+\frac{1}{2}(z/\omega)^2}\right)=\log(1-z/\omega)+z/\omega+\tfrac{1}{2}(z/\omega)^2,$$

and the power series expansion of the logarithm is

$$\log(1-z/w)=-z/\omega-\tfrac{1}{2}(z/\omega)^2-\tfrac{1}{3}(z/\omega)^3-\cdots.$$

Thus the summmand is $\mathcal{O}((z/\omega)^3)$. As discussed in class, this is small enough to make the sum converge nicely. Consequently the infinite product $\sigma(z)$ converges to a holomorphic function on $\mathbb{C}$. The theory of infinite products is covered in many texts. See, for example, *Complex Functions* by Jones and Singerman.)

The *Weierstrass ζ-function*,

$$\zeta:\mathbb{C}\longrightarrow\widehat{\mathbb{C}},$$

emphatically is *not* the Euler–Riemann ζ-function, but instead is

$$\zeta(z)=\log(\sigma(z))'=\frac{\sigma'(z)}{\sigma(z)}=\frac{1}{z}+\sum_{\omega\in\Lambda}{}'\left(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\right).$$

This function has simple poles with residue 1 at the lattice points, analogously to the logarithmic derivative $\pi\cot\pi x$ of $\sin\pi x$, but it isn't quite periodic with respect to Λ. However, let $\Lambda=\omega_1\mathbb{Z}\oplus\omega_2\mathbb{Z}$ where $\omega_1/\omega_2\in\mathcal{H}$. Since the Weierstrass $\wp$-function $\wp=-\zeta'$ is Λ-periodic, the quantities

$$\eta_j=\zeta(z+\omega_j)-\zeta(z),\quad j=1,2$$

are independent of z, i.e., they are lattice constants. Now we have the transformation laws

$$\boxed{\zeta(z+\omega_j)=\zeta(z)+\eta_j,\quad j=1,2.}$$

Consequently,

$$\left(\log\left(\frac{\sigma(z+\omega_j)}{\sigma(z)}\right)\right)'=\zeta(z+\omega_j)-\zeta(z)=\eta_j,\quad j=1,2,$$

and thus for some constant c,

$$\log\left(\frac{\sigma(z+\omega_j)}{\sigma(z)}\right)=\eta_j z+c,\quad j=1,2,$$

or

$$\sigma(z+\omega_j)=\sigma(z)e^{\eta_j z}e^c,\quad j=1,2,$$

To determine e^c, note that the definition of σ shows that σ is odd. Therefore, setting $z=-\omega_j/2$ gives

$$\sigma(\omega_j/2)=-\sigma(\omega_j/2)e^{-\eta_j\sigma_j/2}e^c.$$

And so $e^c=-e^{\eta_j\sigma_j/2}$, giving

$$\boxed{\sigma(z+\omega_j)=-\sigma(z)e^{\eta_j(z+\omega_j/2)},\quad j=1,2.}$$

Incidentally, the lattice constants satisfy the *Legendre relation*,

$$\eta_2\omega_1-\eta_1\omega_2=2\pi i.$$

1.2. **Constructing a function with specified vanishing.** Now let n be a positive integer, and consider a set of data

$$a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}$$

where a_1 through a_n can contain repeats, as can b_1 through b_n, but no a_i and b_j are equal modulo the lattice Λ. Suppose further that

$$\sum_i a_i - \sum_i b_i \in \Lambda.$$

This condition forces $n \geq 2$. We want to define a meromorphic function

$$f : \mathbb{C}/\Lambda \longrightarrow \widehat{\mathbb{C}}$$

with zeros at the a_i, the degree of each zero being the number of times that the corresponding a_i repeats, and similarly for poles at the b_i. Such a function will satisfy the second of our three necessary conditions, $\sum_c \operatorname{ord}_c(f) = 0$, and also the third, $\sum_c \operatorname{ord}_c(f) \cdot c = 0$ in $\mathbb{C}/\Lambda$.

Translating b_n by some lattice element $\lambda \in \Lambda$, which has no effect on the coset $b_n + \Lambda \in \mathbb{C}/\Lambda$, we may assume that in fact

$$\sum_i a_i - \sum_i b_i = 0.$$

Now consider the function

$$f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}, \qquad f(z) = \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)}.$$

This function is meromorphic, and it has the specified zeros and poles. The question is whether f is Λ-periodic. So compute for $j = 1, 2$ that

$$\begin{aligned}
f(z + \omega_j) &= \frac{\prod_i \sigma(z - a_i + \omega_j)}{\prod_i \sigma(z - b_i + \omega_j)} \\
&= \frac{(-1)^n \prod_i \sigma(z - a_i) e^{\eta_j (z - a_i + \omega_j/2)}}{(-1)^n \prod_i \sigma(z - b_i) e^{\eta_j (z - b_i + \omega_j/2)}} \\
&= \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)} \prod_i e^{\eta_j (b_i - a_i)} \\
&= \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)} e^{\eta_j \sum_i (b_i - a_i)} \\
&= \frac{\prod_i \sigma(z - a_i)}{\prod_i \sigma(z - b_i)} \\
&= f(z).
\end{aligned}$$

Thus f is indeed Λ-periodic, giving a meromorphic function on the torus with the specified zeros and poles.

1.3. **Constructing a function with specified principal parts.** Recall that the *Weierstrass $\wp$-function*,

$$\wp : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$$

is

$$\wp(z) = -\zeta'(z) = \frac{1}{z^2} + {\sum_{\omega \in \Lambda}}' \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right).$$

Define also for each integer $k \geq 3$,

$$F_k : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}, \qquad F_k(z) = \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^k}.$$

(Thus $F_k = (-1)^k \wp^{(k-2)}/(k-1)!$.) Recall that the Weierstrass ζ-function has simple poles with residue 1 at the lattice points $\omega \in \Lambda$. More specifically, its Laurent series at 0 is

$$\zeta(z) = \frac{1}{z} + \text{ holomorphic in } z.$$

Similarly, the Weierstrass $\wp$-function Laurent series has a double pole at 0 and Laurent series

$$\wp(z) = \frac{1}{z^2} + \text{ holomorphic in } z,$$

while for $k \geq 3$ the functions F_k have k-fold poles at 0 and Laurent series

$$F_k(z) = \frac{1}{z^k} + \text{ holomorphic in } z.$$

Now let z_1 through z_m be distinct modulo Λ, and consider a set of principal part data

$$\begin{aligned}
P_1(z) &= \frac{c_{1,1}}{z-z_1} + \frac{c_{1,2}}{(z-z_1)^2} + \cdots + \frac{c_{1,n_1}}{(z-z_1)^{n_1}} \\
P_2(z) &= \frac{c_{2,1}}{z-z_2} + \frac{c_{2,2}}{(z-z_2)^2} + \cdots + \frac{c_{2,n_2}}{(z-z_2)^{n_1}} \\
&\vdots \\
P_m(z) &= \frac{c_{m,1}}{z-z_m} + \frac{c_{m,2}}{(z-z_m)^2} + \cdots + \frac{c_{m,n_m}}{(z-z_m)^{n_m}}
\end{aligned}$$

where the coefficients of the minus-first powers sum to zero,

$$c_{1,1} + \cdots + c_{m,1} = 0.$$

These data might describe the principal parts of a meromorphic function on $\mathbb{C}/\Lambda$ at its poles, since the residues of the putative function sum to zero.

The meromorphic function on $\mathbb{C}$ with the desired principal parts is

$$\begin{aligned}
f(z) = c_{1,1}\zeta(z-z_1) &+ c_{1,2}\wp(z-z_1) + \cdots + c_{1,n_1}F_{n_1}(z-z_1) \\
&+ c_{2,1}\zeta(z-z_2) + c_{2,2}\wp(z-z_2) + \cdots + c_{2,n_2}F_{n_2}(z-z_2) \\
&\vdots \\
&+ c_{m,1}\zeta(z-z_m) + c_{m,2}\wp(z-z_m) + \cdots + c_{m,n_m}F_{n_m}(z-z_m).
\end{aligned}$$

More briefly, $f(z) = \sum_{i,j} c_{i,j} F_j(z-z_j)$ where now $F_1 = \zeta$ and $F_2 = \wp$. The question is whether f is Λ-periodic. Since the Weierstrass $\wp$-function and its derivatives are Λ-periodic, the question bears only on the subfunction

$$g(z) = c_{1,1}\zeta(z-z_1) + \cdots + c_{m,1}\zeta(z-z_m) = \sum_{i=1}^m c_{i,1}\zeta(z-z_i).$$

Compute for $j = 1, 2$ that

$$g(z+\omega_j) = \sum_{i=1}^m c_{i,1}\zeta(z-z_i+\omega_j) = \sum_{i=1}^m c_{i,1}(\zeta(z-z_i)+\eta_j) = g(z) + \eta_j \sum_{i=1}^m c_{i,1}.$$

And thus $g(z+\omega_j) = g(z)$ because $\sum_i c_{i,1} = 0$.

2. The field of meromorphic functions on a complex torus

Let Λ be a lattice, and let $\wp$ be its associated Weierstrass function. We show that the field of meromorphic functions on $\mathbb{C}/\Lambda$—or, equivalently, the field of Λ-periodic meromorphic functions on $\mathbb{C}$—is as simple as it possibly could be: it is only the field of rational functions in $\wp$ and $\wp'$,

$$\mathbb{C}(\wp, \wp'),$$

and in fact this field is

$$\mathbb{C}(\wp)[\wp'] = \{f(\wp) + \wp' g(\wp) : f, g \text{ rational functions}\}.$$

So up to isomorphism, the function field is generated by two transcendental quantities over $\mathbb{C}$ that satisfy an algebraic relation,

$$\mathbb{C}(x, y)/\langle y^2 = 4x^3 - g_2 x - g_3\rangle.$$

To establish the desired result, consider any meromorphic function f on $\mathbb{C}/\Lambda$, and introducing two resulting even functions,

$$f_1(z) = \frac{f(z) + f(-z)}{2}, \qquad f_2(z) = \frac{f(z) - f(-z)}{2\wp'(z)}.$$

Then we have the decomposition

$$f(z) = f_1(z) + \wp'(z) f_2(z).$$

This reduces the problem to showing that the field of *even* meromorphic functions on $\mathbb{C}/\Lambda$ is $\mathbb{C}(\wp)$.

So now consider any even meromorphic function f on $\mathbb{C}/\Lambda$, where $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$. Its expansion about 0 is

$$f(z) = \sum_{n \geq \nu_0(f)} a_n z^n, \quad z \text{ near } 0,$$

and so all powers of z in this expansion are even. In particular, the vanishing order $\nu_0(f)$ is even. Similarly, the expansion of f about $\omega_1/2$ is

$$f(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n (z - \tfrac{\omega_1}{2})^n, \quad z \text{ near } 0.$$

Define a related meromorphic function on $\mathbb{C}/\Lambda$,

$$g(z) = f(z + \tfrac{\omega_1}{2}).$$

To see that g is even because f is even and because $\frac{\omega_1}{2}$ is its own inverse in $\mathbb{C}/\Lambda$, compute

$$g(-z) = f(-z + \tfrac{\omega_1}{2}) = f(-z - \tfrac{\omega_1}{2} + \omega_1) = f(-z - \tfrac{\omega_1}{2}) = f(z + \tfrac{\omega_1}{2}) = g(z).$$

Thus the order of g at 0 is even, as shown earlier in this paragraph. But the Laurent expansion of g about 0 is

$$g(z) = \sum_{n \geq \nu_{\omega_1/2}(f)} b_n z^n, \quad z \text{ near } 0.$$

Thus $\nu_{\omega_1/2}(f)$ is even. Similarly, $\nu_{\omega_2/2}(f)$ and $\nu_{(\omega_1+\omega_2)/2}(f)$ are even.

All points of $\mathbb{C}/\Lambda$ come in opposite pairs $\{\pm p\}$, other than (the cosets of) the four points $q = 0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. Given the even meromorphic function f on $\mathbb{C}/\Lambda$, consider the related function

$$\varphi(z) = \prod_p (\wp(z) - \wp(p))^{\nu_p(f)} \prod_q (\wp(z) - \wp(q))^{\nu_q(f)/2}.$$

The first product in the previous display chooses either point of each pair $\{\pm p\}$. The function φ is a rational function in $\wp$. Because $\wp$ takes the values q to order 2, the function φ has the same order of vanishing as f everywhere. Thus their quotient is analytic and doubly periodic, making it constant, and so f is a rational function in $\wp$ as well. This completes the argument.