EXPONENTIAL GROWTH DOMINATES POLYNOMIAL GROWTH

We prove here that

$$\boxed{\lim_{x\to\infty} \frac{x^a}{b^x} = 0 \quad \text{for any } a > 0 \text{ and } b > 1.}$$

For example, $\lim_{x\to\infty} \frac{x^{100000000}}{1.00000001^x} = 0.$

The first thing to show is that

$$\lim_{x\to\infty} \frac{\log x}{x} = 0.$$

The geometry of this formula is shown in figure 1. Recall that $\log x$ is the area under the $y = 1/x$ curve from 1 to x, the shaded area in the figure. On the other hand, x itself is the area of the box in the figure. As the box grows rightward, the shaded area becomes negligable as a portion of the total area.

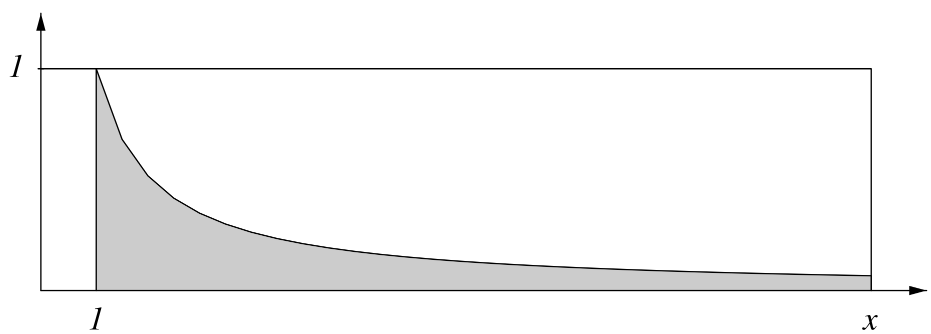

FIGURE 1. $\log x$ as a portion of x

To quantify the argument, see figure 2. Again, $\log x$ is the area from 1 to x. Given any $\varepsilon > 0$ (and also $\varepsilon < 2$), this area is less than the area of the two boxes in the figure, the excess area being shown as lighter gray. Thus, for any $x > 2/\varepsilon$,

$$\log x < \frac{2}{\varepsilon} - 1 + \left(x - \frac{2}{\varepsilon}\right)\frac{\varepsilon}{2} = C + \frac{x\varepsilon}{2} \qquad (\text{where } C = \frac{2}{\varepsilon} - 2).$$

It follows that

$$\frac{\log x}{x} < \frac{C}{x} + \frac{\varepsilon}{2},$$

and so

$$\frac{\log x}{x} < \varepsilon \qquad \text{for all large enough } x.$$

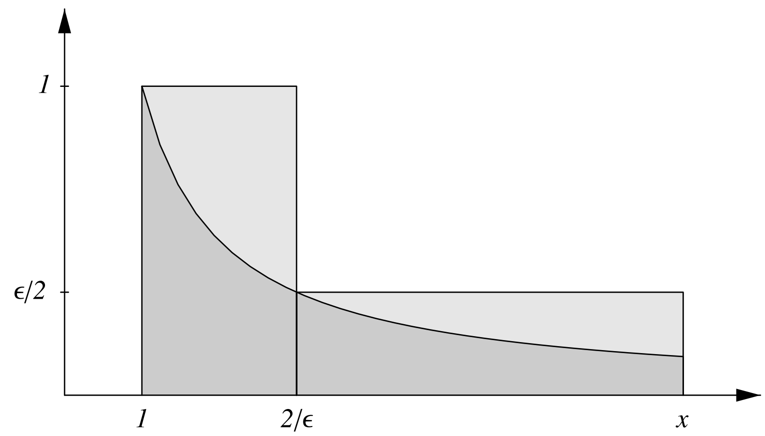

FIGURE 2. $\log x$ is less than the two box-areas

The result follows. Recall that for any given $a > 0$ and $b > 1$, we want to show that

$$\lim_{x\to\infty} \frac{x^a}{b^x} = 0.$$

The idea is that for large enough x we have

$$0 < \frac{\log x}{x} < \frac{\log b}{a+1},$$

i.e.,

$$(a+1)\log x < x \log b,$$

i.e.,

$$x^{a+1} < b^x,$$

i.e.,

$$\frac{x^a}{b^x} < \frac{1}{x}.$$

And since $\lim 1/x = 0$, we are done.