

BEGINNING MODULAR FORMS

This writeup gives first examples of modular forms: Eisenstein series, the discriminant, and the j -function.

1. DEFINITIONS

The *modular group* is the group of 2-by-2 matrices with integer entries and determinant 1,

$$\mathrm{SL}_2(\mathbf{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbf{Z}, ad - bc = 1 \right\}.$$

The modular group is generated by the two matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(Exercise 1). Each element of the modular group is also viewed as an automorphism (invertible self-map) of the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, the fractional linear transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \tau \in \widehat{\mathbf{C}}.$$

This is understood to mean that if $c \neq 0$ then $-d/c$ maps to ∞ and ∞ maps to a/c , and if $c = 0$ then ∞ maps to ∞ . The identity matrix I and its negative $-I$ both give the identity transformation, and more generally each pair $\pm\gamma$ of matrices in $\mathrm{SL}_2(\mathbf{Z})$ gives a single transformation. The group of transformations defined by the modular group is generated by the maps described by the two matrix generators,

$$\tau \mapsto \tau + 1 \quad \text{and} \quad \tau \mapsto -1/\tau.$$

The *upper half plane* is

$$\mathcal{H} = \{\tau \in \mathbf{C} : \mathrm{Im}(\tau) > 0\}.$$

Readers with some background in Riemann surface theory—which is not necessary to read this book—may recognize \mathcal{H} as one of the three simply connected Riemann surfaces, the other two being the plane \mathbf{C} and the sphere $\widehat{\mathbf{C}}$. The formula

$$\mathrm{Im}(\gamma(\tau)) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

(Exercise 2(a)) shows that if $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$ then also $\gamma(\tau) \in \mathcal{H}$, i.e., the modular group maps the upper half plane back to itself. In fact the modular group acts on the upper half plane, meaning that $I(\tau) = \tau$ where I is the identity matrix (as was already noted) and $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$. This last formula is easy to check (Exercise 2(b)).

Definition 1.1. *Let k be an integer. A meromorphic function $f : \mathcal{H} \rightarrow \mathbf{C}$ is **weakly modular of weight k** if*

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \quad \text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \text{ and } \tau \in \mathcal{H}.$$

An argument shows that if this transformation law holds when γ is each of the generators $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then it holds for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. In other words, f is weakly modular of weight k if

$$f(\tau + 1) = f(\tau) \quad \text{and} \quad f(-1/\tau) = \tau^k f(\tau).$$

Weak modularity of weight 0 is simply $\mathrm{SL}_2(\mathbf{Z})$ -invariance, $f \circ \gamma = f$ for all $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. Weak modularity of weight 2 is also natural: complex analysis relies on path integrals of differentials $f(\tau)d\tau$, and $\mathrm{SL}_2(\mathbf{Z})$ -invariant path integration on the upper half plane requires such differentials to be invariant when τ is replaced by any $\gamma(\tau)$. But (Exercise 2(c))

$$d\gamma(\tau) = (c\tau + d)^{-2} d\tau,$$

and so the relation $f(\gamma(\tau))d(\gamma(\tau)) = f(\tau)d\tau$ is

$$f(\gamma(\tau)) = (c\tau + d)^2 f(\tau),$$

giving Definition 1.1 with weight $k = 2$. Weight 2 will play an especially important role later in this book since it is the weight of the modular form in the Modularity Theorem. The weight 2 case also leads inexorably to higher even weights—multiplying two weakly modular functions of weight 2 gives a weakly modular function of weight 4, and so on. Letting $\gamma = -I$ in Definition 1.1 gives $f = (-1)^k f$, showing that the only weakly modular function of any odd weight k is the zero function, but nonzero odd weight examples exist in more general contexts to be developed soon. Another motivating idea for weak modularity is that while it does not make a function f fully $\mathrm{SL}_2(\mathbf{Z})$ -invariant, at least $f(\tau)$ and $f(\gamma(\tau))$ always have the same zeros and poles since the factor $c\tau + d$ on \mathcal{H} has neither.

Modular forms are weakly modular functions that are also holomorphic on the upper half plane and holomorphic at ∞ . To define this last notion, recall that $\mathrm{SL}_2(\mathbf{Z})$ contains the translation matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : \tau \mapsto \tau + 1,$$

for which the factor $c\tau + d$ is simply 1, so that $f(\tau + 1) = f(\tau)$ for every weakly modular function $f : \mathcal{H} \rightarrow \mathbf{C}$. That is, weakly modular functions are \mathbf{Z} -periodic. Let $D = \{q \in \mathbf{C} : |q| < 1\}$ be the open complex unit disk, let $D' = D \setminus \{0\}$, and recall from complex analysis that the \mathbf{Z} -periodic holomorphic map $\tau \mapsto e^{2\pi i\tau} = q$ takes \mathcal{H} to D' . Thus, corresponding to f , the function $g : D' \rightarrow \mathbf{C}$ where $g(q) = f(\log(q)/(2\pi i))$ is well defined even though the logarithm is only determined up to $2\pi i\mathbf{Z}$, and $f(\tau) = g(e^{2\pi i\tau})$. If f is holomorphic on the upper half plane then the composition g is holomorphic on the punctured disk since the logarithm can be defined holomorphically about each point, and so g has a Laurent expansion $g(q) = \sum_{n \in \mathbf{Z}} a_n q^n$ for $q \in D'$. The relation $|q| = e^{-2\pi \mathrm{Im}(\tau)}$ shows that $q \rightarrow 0$ as $\mathrm{Im}(\tau) \rightarrow \infty$. So, thinking of ∞ as lying far in the imaginary direction, define f to be *holomorphic at ∞* if g extends holomorphically to the puncture point $q = 0$, i.e., the Laurent series sums over $n \in \mathbf{N}$. This means that f has a *Fourier expansion*

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i\tau}.$$

Since $q \rightarrow 0$ if and only if $\mathrm{Im}(\tau) \rightarrow \infty$, showing that a weakly modular holomorphic function $f : \mathcal{H} \rightarrow \mathbf{C}$ is holomorphic at ∞ doesn't require computing its

Fourier expansion, only showing that $\lim_{\text{Im}(\tau) \rightarrow \infty} f(\tau)$ exists or even just that $f(\tau)$ is bounded as $\text{Im}(\tau) \rightarrow \infty$.

Definition 1.2. Let k be an integer. A function $f : \mathcal{H} \rightarrow \mathbf{C}$ is a **modular form of weight k** if

- (1) f is holomorphic on \mathcal{H} ,
- (2) f is weakly modular of weight k ,
- (3) f is holomorphic at ∞ .

The set of modular forms of weight k is denoted $\mathcal{M}_k(\text{SL}_2(\mathbf{Z}))$.

It is easy to check that $\mathcal{M}_k(\text{SL}_2(\mathbf{Z}))$ forms a vector space over \mathbf{C} (Exercise 3(a)). Holomorphy at ∞ will make the dimension of this space, and of more spaces of modular forms to be defined in the next section, finite. When f is holomorphic at ∞ it is tempting to define $f(\infty) = g(0) = a_0$, but the next section will show that this doesn't work in a more general context.

The product of a modular form of weight k with a modular form of weight l is a modular form of weight $k + l$ (Exercise 3(b)). Thus the sum

$$\mathcal{M}(\text{SL}_2(\mathbf{Z})) = \bigoplus_{k \in \mathbf{Z}} \mathcal{M}_k(\text{SL}_2(\mathbf{Z}))$$

forms a ring, a so-called graded ring because of its structure as a sum.

2. EXAMPLES

The zero function on \mathcal{H} is a modular form of every weight, and every constant function on \mathcal{H} is a modular form of weight 0. For nontrivial examples of modular forms, let $k > 2$ be an even integer and define the *Eisenstein series of weight k* to be a 2-dimensional analog of the Riemann zeta function $\zeta(k) = \sum_{d=1}^{\infty} 1/d^k$,

$$G_k(\tau) = \sum'_{(c,d)} \frac{1}{(c\tau + d)^k}, \quad \tau \in \mathcal{H},$$

where the primed summation sign means to sum over nonzero integer pairs $(c, d) \in \mathbf{Z}^2 \setminus \{(0, 0)\}$. The sum is absolutely convergent and converges uniformly on compact subsets of \mathcal{H} (Exercise 4(c)), so G_k is holomorphic on \mathcal{H} and its terms may be rearranged. For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbf{Z})$, compute that

$$\begin{aligned} G_k(\gamma(\tau)) &= \sum'_{(c',d')} \frac{1}{\left(c' \left(\frac{a\tau + b}{c\tau + d}\right) + d'\right)^k} \\ &= (c\tau + d)^k \sum'_{(c',d')} \frac{1}{((c'a + d'c)\tau + (c'b + d'd))^k}. \end{aligned}$$

But as (c', d') runs through $\mathbf{Z}^2 \setminus \{(0, 0)\}$, so does $(c'a + d'c, c'b + d'd) = (c', d') \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (Exercise 4(d)), and so the right side is $(c\tau + d)^k G_k(\tau)$, showing that G_k is weakly modular of weight k . Finally, G_k is bounded as $\text{Im}(\tau) \rightarrow \infty$ (Exercise 4(e)), so it is a modular form.

To compute the Fourier series for G_k , continue to let $\tau \in \mathcal{H}$ and begin with the identities

$$(1) \quad \frac{1}{\tau} + \sum_{d=1}^{\infty} \left(\frac{1}{\tau - d} + \frac{1}{\tau + d} \right) = \pi \cot \pi\tau = \pi i - 2\pi i \sum_{m=0}^{\infty} q^m, \quad q = e^{2\pi i\tau}$$

(Exercise 5—the reader who is unhappy with this unmotivated invocation of unfamiliar expressions for a trigonometric function should be reassured that it is a standard rite of passage into modular forms; but also, Exercise 6 provides other proofs, perhaps more natural, of the following formula (2)). Differentiating (1) $k - 1$ times with respect to τ gives for $\tau \in \mathcal{H}$ and $q = e^{2\pi i\tau}$,

$$(2) \quad \sum_{d \in \mathbf{Z}} \frac{1}{(\tau + d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} q^m, \quad k \geq 2.$$

For even $k > 2$,

$$\sum'_{(c,d)} \frac{1}{(c\tau + d)^k} = \sum_{d \neq 0} \frac{1}{d^k} + 2 \sum_{c=1}^{\infty} \left(\sum_{d \in \mathbf{Z}} \frac{1}{(c\tau + d)^k} \right),$$

so again letting ζ denote the Riemann zeta function and using (2) gives

$$\sum'_{(c,d)} \frac{1}{(c\tau + d)^k} = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} m^{k-1} q^{cm}.$$

Rearranging the last expression gives the Fourier expansion

$$G_k(\tau) = 2\zeta(k) + 2 \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad k > 2, \quad k \text{ even}$$

where the coefficient $\sigma_{k-1}(n)$ is the arithmetic function

$$\sigma_{k-1}(n) = \sum_{\substack{m|n \\ m>0}} m^{k-1}.$$

Exercise 7(b) shows that dividing by the leading coefficient gives a series having rational coefficients with a common denominator. This *normalized* Eisenstein series $G_k(\tau)/(2\zeta(k))$ is denoted $E_k(\tau)$. The Riemann zeta function will be discussed further in another handout.

Since the set of modular forms is a graded ring, we can make modular forms out of various sums of products of the Eisenstein series. For example, $\mathcal{M}_8(\mathrm{SL}_2(\mathbf{Z}))$ turns out to be 1-dimensional. The functions $E_4(\tau)^2$ and $E_8(\tau)$ both belong to this space, making them equal up to a scalar multiple and therefore equal since both have leading term 1. Expanding out the relation $E_4^2 = E_8$ gives a relation between the divisor-sum functions σ_3 and σ_7 (Exercise 7(c)),

$$(3) \quad \sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i) \sigma_3(n-i), \quad n \geq 1.$$

The modular forms that, unlike Eisenstein series, have constant term equal to 0 play an important role in the subject.

Definition 2.1. A *cuspidal form* of weight k is a modular form of weight k whose Fourier expansion has leading coefficient $a_0 = 0$, i.e.,

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i\tau}.$$

The set of cuspidal forms is denoted $\mathcal{S}_k(\mathrm{SL}_2(\mathbf{Z}))$.

So a modular form is a cusp form when $\lim_{\text{Im}(\tau) \rightarrow \infty} f(\tau) = 0$. The limit point ∞ of \mathcal{H} is called the *cusp of $\text{SL}_2(\mathbf{Z})$* for geometric reasons that take a bit of work to explain. and a cusp form can be viewed as vanishing at the cusp. The cusp forms $\mathcal{S}_k(\text{SL}_2(\mathbf{Z}))$ form a vector subspace of the modular forms $\mathcal{M}_k(\text{SL}_2(\mathbf{Z}))$, and the graded ring

$$\mathcal{S}(\text{SL}_2(\mathbf{Z})) = \bigoplus_{k \in \mathbf{Z}} \mathcal{S}_k(\text{SL}_2(\mathbf{Z}))$$

is an ideal in $\mathcal{M}(\text{SL}_2(\mathbf{Z}))$ (Exercise 3(c)).

For an example of a cusp form, let

$$g_2(\tau) = 60G_4(\tau), \quad g_3(\tau) = 140G_6(\tau),$$

and define the *discriminant function*

$$\Delta : \mathcal{H} \longrightarrow \mathbf{C}, \quad \Delta(\tau) = (g_2(\tau))^3 - 27(g_3(\tau))^2.$$

Then Δ is weakly modular of weight 12 and holomorphic on \mathcal{H} , and $a_0 = 0$, $a_1 = (2\pi)^{12}$ in the Fourier expansion of Δ (Exercise 7(d)). So indeed $\Delta \in \mathcal{S}_{12}(\text{SL}_2(\mathbf{Z}))$, and Δ is not the zero function. Another writeup will show that in fact $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$ so that the only zero of Δ is at ∞ .

It follows that the *modular function*

$$j : \mathcal{H} \longrightarrow \mathbf{C}, \quad j(\tau) = 1728 \frac{(g_2(\tau))^3}{\Delta(\tau)}$$

is holomorphic on \mathcal{H} . Since the numerator and denominator of j have the same weight, j is $\text{SL}_2(\mathbf{Z})$ -invariant,

$$j(\gamma(\tau)) = j(\tau), \quad \gamma \in \text{SL}_2(\mathbf{Z}), \tau \in \mathcal{H},$$

and in fact it is also called *the modular invariant*. The expansion

$$j(\tau) = \frac{(2\pi)^{12} + \dots}{(2\pi)^{12}q + \dots} = \frac{1}{q} + \dots$$

shows that j has a simple pole at ∞ (and is normalized to have residue 1 at the pole), so it is not quite a modular form. Let ζ_3 denote the complex cube root of unity $e^{2\pi i/3}$. Easy calculations (Exercise 8) show that $g_3(i) = 0$ so that $g_2(i) \neq 0$ and $j(i) = 1728$, and $g_2(\zeta_3) = 0$ so that $g_3(\zeta_3) \neq 0$ and $j(\zeta_3) = 0$. One can further show that

$$g_2(i) = 4\varpi_4^4, \quad \varpi_4 = 2 \int_0^1 \frac{dt}{\sqrt{1-t^4}} = 2\sqrt{\pi} \frac{\Gamma(5/4)}{\Gamma(3/4)}$$

and

$$g_3(\zeta_3) = (27/16)\varpi_3^6, \quad \varpi_3 = 2 \int_0^1 \frac{dt}{\sqrt{1-t^3}} = 2\sqrt{\pi} \frac{\Gamma(4/3)}{\Gamma(5/6)}.$$

Here the integrals are *elliptic integrals*, and Γ is Euler's *gamma function*, to be defined in a separate handout. Finally, Exercise 9 shows that the j -function surjects from \mathcal{H} to \mathbf{C} .

Exercises.

- (1) Let Γ be the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ generated by the two matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n \in \Gamma$ for all $n \in \mathbf{Z}$. Let $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix in $\mathrm{SL}_2(\mathbf{Z})$. Use the identity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b' \\ c & nc + d \end{bmatrix}$$

to show that unless $c = 0$, some matrix $\alpha\gamma$ with $\gamma \in \Gamma$ has bottom row (c, d') with $|d'| \leq |c|/2$. Use the identity

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & -a \\ d & -c \end{bmatrix}$$

to show that this process can be iterated until some matrix $\alpha\gamma$ with $\gamma \in \Gamma$ has bottom row $(0, *)$. Show that in fact the bottom row is $(0, \pm 1)$, and since $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = -I$ it can be taken to be $(0, 1)$. Show that therefore $\alpha\gamma \in \Gamma$ and so $\alpha \in \Gamma$. Thus Γ is all of $\mathrm{SL}_2(\mathbf{Z})$.

- (2) (a) Show that $\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau)/|c\tau + d|^2$ for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.
 (b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$.
 (c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau + d)^2$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.
 (3) (a) Show that the set $\mathcal{M}_k(\mathrm{SL}_2(\mathbf{Z}))$ of modular forms of weight k forms a vector space over \mathbf{C} .
 (b) If f is a modular form of weight k and g is a modular form of weight l , show that fg is a modular form of weight $k + l$.
 (c) Show that $\mathcal{S}_k(\mathrm{SL}_2(\mathbf{Z}))$ is a vector subspace of $\mathcal{M}_k(\mathrm{SL}_2(\mathbf{Z}))$ and that $\mathcal{S}(\mathrm{SL}_2(\mathbf{Z}))$ is an ideal in $\mathcal{M}(\mathrm{SL}_2(\mathbf{Z}))$.
 (4) Let $k \geq 3$ be an integer and let $L' = \mathbf{Z}^2 \setminus \{(0, 0)\}$.
 (a) Show that the series $\sum_{(c,d) \in L'} (\sup\{|c|, |d|\})^{-k}$ converges by considering the partial sums over expanding squares.
 (b) Fix positive numbers A and B and let

$$\Omega = \{\tau \in \mathcal{H} : |\mathrm{Re}(\tau)| \leq A, \mathrm{Im}(\tau) \geq B\}.$$

Prove that there is a constant $C > 0$ such that $|\tau + \delta| > C \sup\{1, |\delta|\}$ for all $\tau \in \Omega$ and $\delta \in \mathbf{R}$.

(c) Use parts (a) and (b) to prove that the series defining $G_k(\tau)$ converges absolutely and uniformly for $\tau \in \Omega$. Conclude that G_k is holomorphic on \mathcal{H} .

(d) Show that for $\gamma \in \mathrm{SL}_2(\mathbf{Z})$, right multiplication by γ defines a bijection from L' to L' .

(e) Use the calculation from (c) to show that G_k is bounded on Ω . From the text and part (d), G_k is weakly modular so in particular $G_k(\tau + 1) = G_k(\tau)$. Show that therefore $G_k(\tau)$ is bounded as $\mathrm{Im}(\tau) \rightarrow \infty$.

- (5) Establish the two formulas for $\pi \cot \pi\tau$ in (1).
 (6) This exercise obtains formula (2) without using the cotangent. Let $f(\tau) = \sum_{d \in \mathbf{Z}} 1/(\tau + d)^k$ for $k \geq 2$ and $\tau \in \mathcal{H}$. Since f is holomorphic (by the method of Exercise 4) and \mathbf{Z} -periodic and since $\lim_{\mathrm{Im}(\tau) \rightarrow \infty} f(\tau) = 0$, there is a Fourier expansion $f(\tau) = \sum_{m=1}^{\infty} a_m q^m = g(q)$ as in the section, where $q = e^{2\pi i\tau}$ and

$$a_m = \frac{1}{2\pi i} \int_{\gamma} \frac{g(q)}{q^{m+1}} dq$$

is a path integral once counterclockwise over a circle about 0 in the punctured disk D' .

(a) Show that

$$a_m = \int_{\tau=0+iy}^{1+iy} f(\tau)e^{-2\pi im\tau} d\tau = \int_{\tau=-\infty+iy}^{+\infty+iy} \tau^{-k} e^{-2\pi im\tau} d\tau \quad \text{for any } y > 0.$$

(b) Let $g_m(\tau) = \tau^{-k} e^{-2\pi im\tau}$, a meromorphic function on \mathbf{C} with its only singularity at the origin. Show that

$$-2\pi i \operatorname{Res}_{\tau=0} g_m(\tau) = \frac{(-2\pi i)^k}{(k-1)!} m^{k-1}.$$

(c) Establish (2) by integrating $g_m(\tau)$ clockwise about a large rectangular path and applying the Residue Theorem. Argue that the integral along the top side goes to a_m and the integrals along the other three sides go to 0.

(d) Let $h : \mathbf{R} \rightarrow \mathbf{C}$ be a function such that the integral $\int_{-\infty}^{\infty} |h(x)| dx$ is finite and the sum $\sum_{d \in \mathbf{Z}} h(x+d)$ converges absolutely and uniformly on compact subsets and is infinitely differentiable. Then the *Poisson summation formula* says that

$$\sum_{d \in \mathbf{Z}} h(x+d) = \sum_{m \in \mathbf{Z}} \hat{h}(m) e^{2\pi imx}$$

where \hat{h} is the *Fourier transform* of h ,

$$\hat{h}(x) = \int_{t=-\infty}^{\infty} h(t) e^{-2\pi ixt} dt.$$

We will not prove this, but the idea is that the left side sum symmetrizes h to a function of period 1 and the right side sum is the Fourier series of the left side since the m th Fourier coefficient is $\int_{t=0}^1 \sum_{d \in \mathbf{Z}} h(t+d) e^{-2\pi imt} dt = \hat{h}(m)$. Letting $h(x) = 1/\tau^k$ where $\tau = x + iy$ with $y > 0$, show that h meets the conditions for Poisson summation. Show that $\hat{h}(m) = e^{-2\pi my} a_m$ with a_m from above for $m > 0$, and that $\hat{h}(m) = 0$ for $m \leq 0$. Establish formula (2) again, this time as a special case of Poisson summation. We will see more Poisson summation and Fourier analysis in connection with Eisenstein series in another handout.

(7) The *Bernoulli numbers* B_k are defined by the formal power series expansion

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Thus they are calculable in succession by matching coefficients in the power series identity

$$t = (e^t - 1) \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} \binom{n}{k} B_k \right) \frac{t^n}{n!}$$

(i.e., the n th parenthesized sum is 1 if $n = 1$ and 0 otherwise) and they are rational. Since the expression

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2} \cdot \frac{e^t + 1}{e^t - 1}$$

is even, it follows that $B_1 = -1/2$ and $B_k = 0$ for all other odd k . The Bernoulli numbers will be motivated, discussed, and generalized in another handout.

- (a) Show that $B_2 = 1/6$, $B_4 = -1/30$, and $B_6 = 1/42$.
 (b) Use the expressions for $\pi \cot \pi\tau$ from the section to show

$$1 - 2 \sum_{k=1}^{\infty} \zeta(2k) \tau^{2k} = \pi \tau \cot \pi\tau = \pi i \tau + \sum_{k=0}^{\infty} B_k \frac{(2\pi i \tau)^k}{k!}.$$

Use these to show that for $k \geq 2$ even, the Riemann zeta function satisfies

$$2\zeta(k) = -\frac{(2\pi i)^k}{k!} B_k,$$

so in particular $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and $\zeta(6) = \pi^6/945$. Also, this shows that the normalized Eisenstein series of weight k

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

has rational coefficients with a common denominator.

- (c) Equate coefficients in the relation $E_8(\tau) = E_4(\tau)^2$ to establish formula (3).
 (d) Show that $a_0 = 0$ and $a_1 = (2\pi)^{12}$ in the Fourier expansion of the discriminant function Δ from the text.
 (8) Recall that ζ_3 denotes the complex cube root of unity $e^{2\pi i/3}$. Show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\zeta_3) = \zeta_3 + 1$ so that by periodicity $g_2(\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\zeta_3)) = g_2(\zeta_3)$. Show that by modularity also $g_2(\begin{bmatrix} a & b \\ c & d \end{bmatrix}(\zeta_3)) = \zeta_3^4 g_2(\zeta_3)$ and therefore $g_2(\zeta_3) = 0$. Conclude that $g_3(\zeta_3) \neq 0$ and $j(\zeta_3) = 0$. Argue similarly to show that $g_3(i) = 0$, $g_2(i) \neq 0$, and $j(i) = 1728$.
 (9) This exercise shows that the modular invariant $j : \mathcal{H} \rightarrow \mathbf{C}$ is a surjection. Suppose that $c \in \mathbf{C}$ and $j(\tau) \neq c$ for all $\tau \in \mathcal{H}$. Consider the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{j'(\tau) d\tau}{j(\tau) - c}$$

where γ is the contour containing an arc of the unit circle from $(-1+i\sqrt{3})/2$ to $(1+i\sqrt{3})/2$, two vertical segments up to any height greater than 1, and a horizontal segment. By the Argument Principle the integral is 0. Use the fact that j is invariant under $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to show that the integrals over the two vertical segments cancel. Use the fact that j is invariant under $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to show that the integrals over the two halves of the circular arc cancel. For the integral over the remaining piece of γ make the change of coordinates $q = e^{2\pi i\tau}$, remembering that $j'(\tau)$ denotes derivative with respect to τ and that $j(\tau) = 1/q + \dots$, and compute that it equals 1. This contradiction shows that $j(\tau) = c$ for some $\tau \in \mathcal{H}$ and j surjects.