COMPLEX POWER SERIES

A complex power series is a formal infinite complex polynomial in an indeterminate z, determined by a list of coefficients $a_n \in \mathbb{C}$ for $n = 0, 1, 2, \ldots$ and a center $c \in \mathbb{C}$,

$$p(z) = \sum_{n=0}^{\infty} a_n (z-c)^n = a_0 + a_1(z-c) + a_2(z-c)^2 + \cdots .$$

Because this sum is infinite, substituting a specific value $z_o \in \mathbb{C}$ for z produces an infinite series $p(z_o)$ of complex numbers.

- $p(z_o)$ may converge, meaning that $\lim_{n\to\infty} \sum_{k=0}^{n} a_k (z-c)^k$ exists in $\mathbb{C}$. Such convergence can be fragile, in the sense that rearranging the terms of $p(z_o)$ can give a new complex series that converges to a different value or diverges. The value of $p(z_o)$ is *fragile,* not persisting under credible-seeming series manipulations.
- $p(z_o)$ may converge absolutely, meaning that $\lim_{n\to\infty} \sum_{k=0}^{n} |a_k (z-c)^k|$ exists in $\mathbb{R}_{\geq 0}$. Consequently $p(z_o)$ converges and further all of its rearrangements converge to the same value. Now the value of $p(z_o)$ is *robust.*
- $p(z_o)$ may diverge.

The power series $p(z)$ has a radius of convergence $R \in [0, +\infty]$ (possibly $R = +\infty$) such that

- $p(z)$ converges *absolutely* on the open disk $\{z : |z - c| < R\}$, and further the absolute convergence is *uniform on compact subsets* of this disk. Consequently $p(z)$ converges robustly on the open disk and its convergence is also uniform on compact subsets. In particular, $p(z)$ is continuous on each compact subset of the open disk and therefore on the entire open disk.
- $p(z)$ diverges on the complement $\{z : |z - c| > R\}$ of the closed disk.

We make no assertion about the behavior of $p(z)$ for values z such that $|z - c| = R$. The interested reader could consider the series $\sum_{n=1}^{\infty} z^n/n^e$ for various values of the fixed exponent e and for z such that $|z| = 1$.

Contents

1. Complex series: consequences of absolute convergence

1.1. Absolute convergence implies convergence. We show:

Let $\{c_n\}$ be a sequence of complex terms.
If $\sum_n |c_n|$ converges then $\sum_n c_n$ converges.

Initially assume that $\{c_n\}$ is a real sequence. The condition

$$-|c_n| \leq c_n \leq |c_n| \quad \text{for all } n$$

gives

$$0 \leq c_n + |c_n| \leq 2|c_n| \quad \text{for all } n$$

and so $\sum_n (c_n + |c_n|)$ converges by comparison to $\sum_n 2|c_n| = 2\sum_n |c_n|$. Consequently the series $\sum_n c_n$ is the difference of two convergent series and hence convergent,

$$\sum_n c_n = \sum_n (c_n + |c_n| - |c_n|) = \sum_n (c_n + |c_n|) - \sum_n |c_n|.$$

Now let $\{c_n\}$ be any complex sequence. The conditions $|\mathrm{Re}(c_n)| \leq |c_n|$ and $|\mathrm{Im}(c_n)| \leq |c_n|$ for all n show that $\sum_n |\mathrm{Re}(c_n)|$ and $\sum_n |\mathrm{Im}(c_n)|$ converge by comparison to $\sum_n |c_n|$. As just shown, $\sum_n \mathrm{Re}(c_n)$ and $\sum_n \mathrm{Im}(c_n)$ converge in consequence. Therefore the series

$$\sum_n c_n = \sum_n (\mathrm{Re}(c_n) + i\mathrm{Im}(c_n)) = \sum_n \mathrm{Re}(c_n) + i \sum_n \mathrm{Im}(c_n)$$

converges.

The comparison test, used in the previous paragraph, relies on the completeness of the real number system. If we are comfortable working directly with the completeness of $\mathbb{C}$ then the argument can be made more concisely, as follows. Because $\sum_n |c_n|$ converges, its sequence of partial sums is Cauchy. This means that its (nonnegative) partial tails,

$$|c_N| + |c_{N+1}| + \cdots + |c_M|$$

get small as N and M grow. It follows by the triangle inequality that the absolute values of the partial tails of $\sum_n c_n$,

$$\left|c_N + c_{N+1} + \cdots + c_M\right|$$

get small as N and M grow, and so the sequence of partial sums of $\sum_n c_n$ is Cauchy. Because $\mathbb{C}$ is complete, $\sum_n c_n$ converges.

1.2. **Absolute convergence implies rearrangeability.** We show:

Let $\{c_n\}$ be a sequence of complex terms.
If $\sum_n |c_n|$ converges, so that $\sum_n c_n$ converges,
then any rearrangement of $\sum_n c_n$ converges to the same value.

Indeed, consider an absolutely summable sequence

$$\{c_n\} = \{c_0, c_1, c_2, \dots\}, \quad \sum_n |c_n| \text{ converges},$$

and consider a rearrangement of it,

$$\{d_m\} = \{d_0, d_1, d_2, \dots\}, \quad \begin{pmatrix} \text{each } d_m \text{ is } c_{k_m} \text{ for a unique index } k_m \\ \text{each } c_k \text{ is } d_{m_k} \text{ for a unique index } m_k \end{pmatrix}.$$

Let the sequences of partial sums of the sequences $\{c_n\}$, $\{|c_k|\}$, and $\{d_m\}$ be denoted $\{s_n\}$, $\{S_k\}$, and $\{t_m\}$. Thus

$$\{S_k\} \text{ converges to some value } S,$$

from which we now know that

$$\{s_n\} \text{ converges to some value } s,$$

and we want to show that therefore

$$\{t_m\} \textit{ also} \text{ converges, to } \textit{the same} \text{ value } s.$$

Let $\varepsilon > 0$ be given. Because $\{S_k\}$ converges increasingly to S and $\{s_n\}$ converges to s, there exists an index N such that

$$S - S_N < \varepsilon/2 \quad \text{and} \quad |s_N - s| < \varepsilon/2.$$

There exists a natural number $M \geq N$, specifically, $M = \max\{m_0, m_1, \dots, m_N\}$ where $d_{m_k} = c_k$ for $k = 0, \dots, N$, such that

$$\{d_0, d_1, \dots, d_M\} \quad \text{contains} \quad \{c_0, c_1, \dots, c_N\}.$$

So for any index $m \geq M$, $t_m - s_N$ is a finite sum of terms c_k with $k > N$,

$$t_m - s_N = \sum_{k \in K} c_k, \quad K \subset \mathbb{Z}_{>N} \text{ finite}.$$

By the triangle inequality,

$$|t_m - s| \leq |t_m - s_N| + |s_N - s| \leq \sum_{k \in K} |c_k| + |s_N - s| \leq |S - S_N| + |s_N - s|.$$

Because $S - S_N < \varepsilon/2$ and $|s_N - s| < \varepsilon/2$ from above, this gives

$$|t_m - s| < \varepsilon \quad \text{for all } m \geq M.$$

Thus $\{t_m\}$ converges to s. This is the desired result.

2. Complex power series: disk of convergence

2.1. **Existence of radius.** Consider a complex power series, now centered at 0 without loss of generality,

$$p(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots.$$

The value of $p(z)$ for $z = 0$ is simply a_0, but for any other $z \in \mathbb{C}$, $p(z)$ could conceivably converge or diverge.

We show that for some extended nonnegative real number R, including the possibility $R = +\infty$, $p(z)$ converges absolutely on the open disk of radius R, the absolute convergence uniform on compact subsets of the open disk, so that also $p(z)$ converges on the open disk and converges uniformly on compact subsets of the open disk; and $p(z)$ diverges on the complement of the closed disk of radius R. That is,

- $p(z)$ converges absolutely on $\{|z| < R\}$, and this absolute convergence is uniform on compact subsets
- so $p(z)$ converges robustly if $|z| < R$ and this convergence is uniform on compact subsets
- and $p(z)$ diverges if $|z| > R$.

If $R = 0$ then the assertion is that $p(z)$ diverges for all nonzero z, and if $R = +\infty$ then it is that $p(z)$ converges absolutely for all z. No assertion is made about the convergence of $p(z)$ if $|z| = r$, i.e., if z lies on the boundary circle of the disk.

First we show:

If w is a complex number such that $p(w)$ converges
and if $|z| < |w|$
then $p(z)$ converges absolutely.

Indeed, the absolute value of the nth term of $p(z)$ satisfies (noting that $|w| > 0$ because $|w| > |z| \geq 0$)

$$|a_n z^n| = |a_n w^n z^n / w^n| = |a_n w^n| r^n \quad \text{where } r = |z|/|w| < 1.$$

Because $p(w) = \sum_{n=0}^{\infty} a_n w^n$ converges, its terms $a_n w^n$ go to zero and therefore are bounded by some positive constant c, and so

$$|a_n z^n| < cr^n \quad \text{where } r < 1.$$

Thus $p(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely by comparison to $\sum_{n=0}^{\infty} cr^n$.

Further we show:

The absolute convergence of p
is uniform on every compact subset of $\{z : |z| < |w|\}$.

Let K be such a set. The constant $r = \max_{z \in K} |z|/|w|$ satisfies $|z|/|w| \leq r < 1$ for all $z \in K$. Now given $\varepsilon > 0$ there exists n_o, dependent on w but independent of $z \in K$, such that $|a_m w^m| < (1-r)\varepsilon$ for all $m \geq n_o$. Thus for every $n \geq n_o$, simultaneously for all $z \in K$,

$$\sum_{m \geq n_o} |a_m z^m| \leq \sum_{m \geq n_o} |a_m w^m| r^m < (1-r)\varepsilon \sum_{m \geq n_o} r^m \leq \varepsilon.$$

This says that the absolute convergence of $p(z)$ is uniform on K.

Immediately, $p(z)$ converges on $\{|z| < |w|\}$, and again the convergence is uniform on compact subsets because

$$\left| \sum_{m \geq n_o} a_m z^m \right| \leq \sum_{m \geq n_o} |a_m z^m|.$$

Because p is the uniform limit of continuous functions on K, these functions being its polynomial truncations, and because K is compact, we know that p is continuous on K. And because each z such that $|z| < |w|$ lies in a compact subset of $\{z : |z| < |w|\}$, in fact p is continuous on this entire open disk.

If $p(z)$ diverges and $|z| < |w|$ then $p(w)$ diverges. This is because the convergence of $p(w)$ would make $p(z)$ converge absolutely and therefore converge, which it doesn't.

Now, if $p(z)$ converges only for $z = 0$ then our claimed main result holds with $R = 0$, and if $p(z)$ converges for all $z \in \mathbb{C}$ then it holds with $R = +\infty$. So we assume that $p(z)$ converges for some nonzero z and that $p(w)$ diverges for some w. Consequently $p(x)$ converges for every nonnegative real number x such that $x < |z|$ and diverges for every nonnegative real number x such that $x > |w|$. With this in mind, define

$$X = \{x \in \mathbb{R}_{\geq 0} : p(x) \text{ converges}\},$$

a nonempty subset of $\mathbb{R}$ that is bounded from above, and then define

$$R = \sup X.$$

Thus if $0 \leq r < R$ then $p(r)$ converges, and if $r > R$ then $p(r)$ diverges. To spell these two facts out a bit more, reason as follows:

- If $0 \leq r < R$ then r is less than the least upper bound of X and so is not an upper bound of X; thus $p(x)$ converges for some $x > r$, and consequently $p(r)$ converges. (In fact $p(r)$ converge absolutely, but we don't need this fact here.)
- If $r > R$ then r exceeds an upper bound of X and so cannot lie in X; thus $p(r)$ diverges.

For any z such that $|z| < R$, also $|z| < r < R$ where $r = (|z| + R)/2$. Because $0 \leq r < R$ it follows that $p(r)$ converges, and then because $|z| < r$ it follows in turn that $p(z)$ converges absolutely.

And for any z such that $|z| > R$, also $|z| > r > R$ where $r = (|z|+R)/2$. Because $r > R$ it follows that $p(r)$ diverges, and then because $|z| > r$ it follows in turn that $p(z)$ diverges.

An explicit formula for the radius of convergence exists, but it involves a technical idea called the *limit superior.* The student who wants to pursue this can find an exposition at

`https://people.reed.edu/~jerry/311/limsup.pdf` .

2.2. **Persistence of radius.** We show that the two series

$$p(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad P(z) = \sum_{n=0}^{\infty} n a_n z^n$$

have the same radius of convergence, despite the coefficients of P being bigger in absolute value. Let D_p and D_P respectively denote the open disks of absolute convergence of these two series. We want to show that $D_p = D_P$. Further let $\overline{D}_p$ and $\overline{D}_P$ denote the corresponding closed disks. It suffices to show that $D_P \subset \overline{D}_p$ and $D_p \subset \overline{D}_P$, because the first of these implies that $D_P \subset D_p$ since D_p is the largest open disk in $\overline{D}_p$, and similarly the second implies that $D_p \subset D_P$, and the containments $D_P \subset D_p$ and $D_p \subset D_P$ together give $D_p = D_P$.

Consider any point z in D_P. Thus $P(z)$ converges absolutely, which is to say that $\sum_{n=0}^{\infty} n|a_n z^n|$ converges. By comparison, $\sum_{n=0}^{\infty} |a_n z^n|$ converges, which is to say that $p(z)$ converges absolutely. Thus z either lies in D_p or lies on its boundary circle, i.e., z lies in $\overline{D}_p$. So this paragraph has shown that $D_P \subset \overline{D}_p$.

The other containment is the substance of the matter. Consider any point z in D_p. Because D_p is an open disk, for some $\delta > 0$ also $(1+\delta)z \in D_p$. Thus $p((1+\delta)z)$ converges absolutely, which is to say that $\sum_{n=0}^{\infty}(1+\delta)^n|a_n z^n|$ converges. For each n,

$$(1+\delta)^n = 1 + n\delta + \cdots > n\delta.$$

So by comparison, $\sum_{n=0}^{\infty} n\delta|a_n z^n|$ converges, and therefore $\sum_{n=0}^{\infty} n|a_n z^n|$ converges, which is to say that $P(z)$ converges absolutely. Thus z either lies in D_P or lies on its boundary circle, i.e., z lies in $\overline{D}_P$. So this paragraph has shown that $D_p \subset \overline{D}_P$. The argument is complete.

With the result in hand, we note that $\sum_{n=2}^{\infty} a_n n^2 z^{n-2}$ has the same radius of convergence as $\sum_{n=0}^{\infty} a_n n^2 z^n$, which has the same radius of convergence as $\sum_{n=0}^{\infty} a_n n z^n$, which has the same radius of convergence as $\sum_{n=0}^{\infty} a_n z^n$.

The persistence result of this section is immediate from the explicit limit superior formula for the radius of convergence. Here the point is that $\lim_n(n^{1/n}) = 1$; this holds because for any $\varepsilon > 0$, for all large enough n we have $1 < \frac{n-1}{2}\varepsilon^2$ and then, using the binomial coefficient $\binom{n}{2} = \frac{n(n-1)}{2}$ and the binomial theorem $(1+\varepsilon)^n = \sum_{k=0}^n \binom{n}{k}\varepsilon^k$ with the sum including the term $\binom{n}{2}\varepsilon^2$,

$$1 < \frac{n-1}{2}\varepsilon^2 \implies n < \binom{n}{2}\varepsilon^2 \implies n < (1+\varepsilon)^n \implies n^{1/n} < 1+\varepsilon.$$