GEOMETRY OF THE CAUCHY–RIEMANN EQUATIONS

The usual picture-explanations given to interpret the divergence and the curl are not entirely satisfying. Working with the polar coordinate system further quantifies the ideas and makes them more coherent by applying to both operators in the same way. In the process, the Cauchy–Riemann equations emerge with no reference to complex analysis.

Let $A \subset \mathbf{R}^2$ be an open set that contains the origin, and let F be a continuous vector field on A that is stationary at the origin,

$$F = (F_1, F_2) : A \longrightarrow \mathbf{R}^2, \qquad F(\mathbf{0}) = \mathbf{0}.$$

(Rather than study the divergence and the curl of a vector field F at a general point p, we have normalized p to be $\mathbf{0}$ by prepending a translation of the domain, and since the divergence and the curl are differential operators and hence insensitive to constants, we also may normalize $F(\mathbf{0})$ to $\mathbf{0}$ by postpending a translation of the range.) At any point other than the origin, F resolves into a radial component and an angular component. Specifically,

$$F = F_r + F_\theta,$$

where

$$\begin{aligned} F_r &= f_r \hat{r}, \qquad & f_r &= F \cdot \hat{r}, \qquad & \hat{r} &= (\cos\theta, \sin\theta) = (x,y)/|(x,y)|, \\ F_\theta &= f_\theta \hat{\theta}, \qquad & f_\theta &= F \cdot \hat{\theta}, \qquad & \hat{\theta} &= \hat{r}^\times = (-\sin\theta, \cos\theta) = (-y,x)/|(x,y)|. \end{aligned}$$

(The unary cross product $(x,y)^\times = (-y,x)$ in $\mathbf{R}^2$ rotates vectors 90 degrees counterclockwise.) Here f_r is positive if F_r points outward and negative if F_r points inward, and f_θ is positive if F_θ points counterclockwise and negative if F_θ points clockwise. Since $F(\mathbf{0}) = \mathbf{0}$, the resolution of F into radial and angular components extends continuously to the origin, $f_r(\mathbf{0}) = f_\theta(\mathbf{0}) = 0$, so that $F_r(\mathbf{0}) = F_\theta(\mathbf{0}) = \mathbf{0}$ even though $\hat{r}$ and $\hat{\theta}$ are undefined at the origin.

The goal of this writeup is to express the divergence and the curl of F at the origin in terms of the polar coordinate system derivatives that seem naturally suited to describe them, the radial derivative of the (scalar) radial component of F,

$$D_r f_r(\mathbf{0}) = \lim_{r \to 0^+} \frac{f_r(r\cos\theta, r\sin\theta)}{r},$$

and the radial derivative of the (scalar) angular component of F,

$$D_r f_\theta(\mathbf{0}) = \lim_{r \to 0^+} \frac{f_\theta(r\cos\theta, r\sin\theta)}{r}.$$

However, matters aren't as simple as one might hope. If the (vector) radial and angular components F_r and F_θ are differentiable at the origin then so is their sum F, but the converse is not true. So first we need sufficient conditions for the converse, i.e., sufficient conditions for the components to be differentiable. Necessary conditions are always easier to find, so Proposition 1 will do so, and then Proposition 2 will show that the necessary conditions are also sufficient. The conditions in question are the Cauchy–Riemann equations,

$$D_1F_1(\mathbf{0}) = D_2F_2(\mathbf{0}),$$
$$D_1F_2(\mathbf{0}) = -D_2F_1(\mathbf{0}).$$

When the Cauchy–Riemann equations hold, we can describe the divergence and the curl of F at the origin in polar terms, as desired. This will be the content of Theorem 3.

Before we proceed to the details, a brief geometric discussion of the Cauchy–Riemann equations may be helpful. The equation $D_1F_1 = D_2F_2$ describes the left side of figure 1, in which the radial component of F on the horizontal axis is growing at the same rate as the radial component on the vertical axis. Similarly, the equation $D_2F_1 = -D_1F_2$ describes the right side of the figure, in which the angular component on the vertical axis is growing at the same rate as the angular component on the horizontal axis. Combined with differentiability at the origin, these two conditions will imply that moving outward in any direction, the radial component of F is growing at the same rate as it is on the axes, and similarly for the angular component. Thus the two limits that define the radial derivatives of the radial and angular components of F at $\mathbf{0}$ (these were displayed in the previous paragraph) are independent of θ. An example of this situation, with radial and angular components both present, is shown in figure 2. From the perspective of complex analysis, we recognize the figure as a depiction of the function $f(z) = re^{i\theta}z$ for some fixed r and θ.

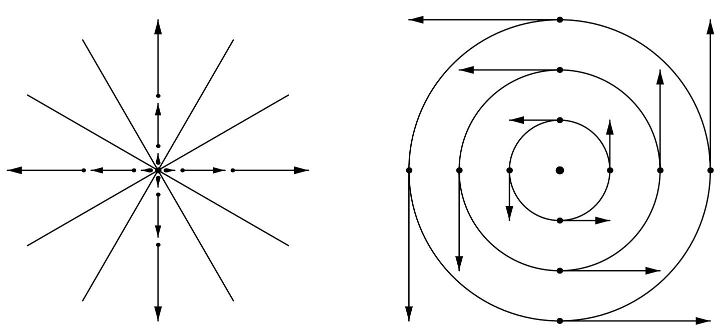

FIGURE 1. Geometry of the Cauchy–Riemann equations individually

As mentioned, the necessity of the Cauchy–Riemann equations is the natural starting point.

Proposition 1: Polar Differentiability Implies Differentiability and the Cauchy–Riemann Equations. *Let $A \subset \mathbf{R}^2$ be an open set that contains the origin, and let F be a continuous vector field on A that is stationary at the origin,*

$$F = (F_1, F_2) : A \longrightarrow \mathbf{R}^2, \qquad F(\mathbf{0}) = \mathbf{0}.$$

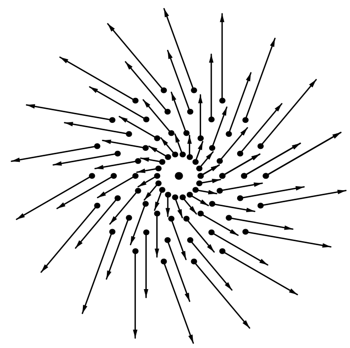

FIGURE 2. Geometry of the Cauchy–Riemann equations together

Assume that the radial and angular components F_r and F_θ of F are differentiable at the origin. Then F is differentiable at the origin, and the Cauchy–Riemann equations hold at the origin.

For example, the vector field $F(x, y) = (x, 0)$ is differentiable at the origin, but since $D_1F_1(\mathbf{0}) = 1$ and $D_2F_2(\mathbf{0}) = 0$, it does not satisfy the Cauchy–Riemann equations, and so the derivatives of the radial and angular components of F at the origin do not exist.

Proof. As already noted, the differentiability of F at the origin is immediate. To begin establishing the Cauchy–Riemann equations, consider the radial component of F,

$$F_r(x,y) = \begin{cases} f_r(x,y)\frac{(x,y)}{|(x,y)|} & \text{if } (x,y) \neq \mathbf{0}, \\ \mathbf{0} & \text{if } (x,y) = \mathbf{0} \end{cases}$$
$$= \begin{cases} g_r(x,y)(x,y) & \text{if } (x,y) \neq \mathbf{0}, \\ \mathbf{0} & \text{if } (x,y) = \mathbf{0}, \end{cases}$$

where

$$g_r(x,y) = \frac{f_r(x,y)}{|(x,y)|} \quad \text{for } (x,y) \neq \mathbf{0}.$$

The first component function of F_r vanishes at $\mathbf{0}$, i.e., $F_{r,1}(\mathbf{0}) = 0$. Also the first component function of F_r vanishes on the y-axis away from the origin because F_r is radial, and so $D_2F_{r,1}(\mathbf{0}) = 0$ as well. Thus the condition that $F_{r,1}$ is differentiable at $\mathbf{0}$ is

$$\lim_{(h,k)\to\mathbf{0}} \frac{|F_{r,1}(h,k) - hD_1F_{r,1}(\mathbf{0})|}{|(h,k)|} = 0,$$

But $F_{\theta,1}$ vanishes on the x-axis since F_θ is angular, and so $D_1F_{\theta,1}(\mathbf{0}) = 0$, giving $D_1F_{r,1}(\mathbf{0}) = D_1F_1(\mathbf{0})$. Also $F_{r,1}(h,k) = h\,g_r(h,k)$ away from the origin, so that the previous condition becomes

$$\lim_{(h,k)\to\mathbf{0}} \frac{|h|\,|g_r(h,k) - D_1F_1(\mathbf{0})|}{|(h,k)|} = 0,$$

A similar argument using the second component function shows that

$$\lim_{(h,k)\to\mathbf{0}} \frac{|k|\,|g_r(h,k) - D_2F_2(\mathbf{0})|}{|(h,k)|} = 0.$$

Let $(h,k)\to\mathbf{0}$ along the line $h=k$ to see that

$$D_1F_1(\mathbf{0}) = D_2F_2(\mathbf{0})$$

since both are $\lim_{h\to 0} g_r(h,h)$. This is the first Cauchy–Riemann equation at the origin, but we show a bit more, to be used in the proof of Theorem 3. Add the previous two displayed limits to get

$$\lim_{(h,k)\to\mathbf{0}} \frac{(|h|+|k|)\,|g_r(h,k) - D_1F_r(\mathbf{0})|}{|(h,k)|} = 0,$$

and since $|(h,k)| \le |h|+|k|$, it follows that

$$\lim_{(h,k)\to\mathbf{0}} |g_r(h,k) - D_1F_r(\mathbf{0})| = 0.$$

That is,

$$\lim_{(h,k)\to\mathbf{0}} g_r(h,k) = D_1F_r(\mathbf{0}) = D_2F_{r,2}(\mathbf{0}).$$

Next consider the radial component of the vector field $-F^\times = f_\theta\hat{r} - f_r\hat{\theta}$,

$$(-F^\times)_r(x,y) = \begin{cases} g_\theta(x,y)(x,y) & \text{if } (x,y)\ne\mathbf{0},\\ \mathbf{0} & \text{if } (x,y)=\mathbf{0},\end{cases}$$

where

$$g_\theta(x,y) = \frac{f_\theta(x,y)}{|(x,y)|} \quad \text{for } (x,y)\ne\mathbf{0}.$$

This radial component is differentiable at the origin since it is a rotation of the angular component of the original F, so as just argued,

$$\lim_{(x,y)\longrightarrow\mathbf{0}} g_\theta(x,y) = -D_1F_1^\times(\mathbf{0}) = -D_2F_2^\times(\mathbf{0}).$$

But

$$-D_1F_1^\times(\mathbf{0}) = D_1F_2(\mathbf{0}) \qquad \text{and} \qquad -D_2F_2^\times(\mathbf{0}) = -D_2F_1(\mathbf{0}),$$

and this gives the second Cauchy–Riemann equation at the origin. □

Also as mentioned, the converse to Proposition 1 holds too.

Proposition 2: Differentiability and the Cauchy–Riemann Equations Imply Polar Differentiability. *Let $A\subset\mathbf{R}^2$ be an open set that contains the origin, and let F be a continuous vector field on A that is stationary at the origin,*

$$F=(F_1,F_2): A\longrightarrow\mathbf{R}^2, \qquad F(\mathbf{0})=\mathbf{0}.$$

Assume that F is differentiable at the origin, and assume that the Cauchy–Riemann equations hold at the origin. Then the radial and angular components F_r and F_θ are differentiable at the origin.

Proof. Let $a = D_1F_1(\mathbf{0})$ and let $b = D_1F_2(\mathbf{0})$. By the Cauchy–Riemann equations, also $a = D_2F_2(\mathbf{0})$ and $b = -D_2F_1(\mathbf{0})$, so that the Jacobian matrix of F at $\mathbf{0}$ is

$$F'(\mathbf{0}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The condition that F is differentiable at $\mathbf{0}$ is

$$\lim_{(h,k)\to\mathbf{0}} \frac{|F(h,k)-(ah-bk,bh+ak)|}{|(h,k)|}=0.$$

Decompose the quantity whose absolute value is the numerator into radial and angular components,

$$F(h,k)-(ah-bk,bh+ak)=\big(F_r(h,k)-(ah,ak)\big)+\big(F_\theta(h,k)-(-bk,bh)\big)$$

Since the direction vectors $\hat{r}=(h,k)/|(h,k)|$ and $\hat{\theta}=(-k,h)/|(h,k)|$ are orthogonal, and

$$F_r(h,k)-(ah,ak)\parallel\hat{r} \qquad \text{and} \qquad F_\theta(h,k)-(-bk,bh)\parallel\hat{\theta},$$

it follows that

$$|F_r(h,k)-(ah,ak)|\le|F(h,k)-(ah-bk,bh+ak)|$$

and

$$|F_\theta(h,k)-(-bk,bh)|\le|F(h,k)-(ah-bk,bh+ak)|.$$

Therefore,

$$\lim_{(h,k)\to\mathbf{0}} \frac{|F_r(h,k)-(ah,ak)|}{|(h,k)|}=0$$

and

$$\lim_{(h,k)\to\mathbf{0}} \frac{|F(h,k)-(-bk,bh)|}{|(h,k)|}=0.$$

That is, F_r and F_θ are differentiable at the origin with respective Jacobian matrices

$$F_r'(\mathbf{0})=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \qquad \text{and} \qquad F_\theta'(\mathbf{0})=\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}.$$

This completes the proof. □

Now we can return to the divergence and the curl.

Theorem 3: Divergence and Curl in Polar Coordinates. *Let $A\subset\mathbf{R}^2$ be a region of $\mathbf{R}^2$ containing the origin, and let F be a continuous vector field on A that is stationary at the origin,*

$$F=(F_1,F_2):A\longrightarrow\mathbf{R}^2, \qquad F(\mathbf{0})=\mathbf{0}.$$

Assume that F is differentiable at the origin and that the Cauchy–Riemann equations hold at the origin. Then the radial derivatives of the radial and angular components of F at the origin,

$$D_r f_r(\mathbf{0})=\lim_{r\to0^+}\frac{f_r(r\cos\theta,r\sin\theta)}{r}$$

and

$$D_r f_\theta(\mathbf{0})=\lim_{r\to0^+}\frac{f_\theta(r\cos\theta,r\sin\theta)}{r},$$

both exist independently of how θ behaves as r shrinks to 0. Furthermore, the divergence of F at the origin is twice the radial derivative of the radial component,

$$(\operatorname{div}F)(\mathbf{0})=2D_r f_r(\mathbf{0}),$$

and the curl of F at the origin is twice the radial derivative of the angular component,

$$(\operatorname{curl}F)(\mathbf{0})=2D_r f_\theta(\mathbf{0}).$$

Proof. By Proposition 1, the angular and radial components of F are differentiable at the origin, so that the hypotheses of Proposition 2 are met. The first limit in the statement of the theorem was calculated in the proof of Proposition 1.

$$D_r f_r(\mathbf{0}) = \lim_{(x,y)\to\mathbf{0}} \frac{f_r(x,y)}{|(x,y)|} = \lim_{(x,y)\to\mathbf{0}} g_r(x,y) = D_1F_1(\mathbf{0}) = D_2F_2(\mathbf{0}).$$

This makes the formula for the divergence immediate,

$$(\operatorname{div} F)(\mathbf{0}) = D_1F_1(\mathbf{0}) + D_2F_2(\mathbf{0}) = 2D_r f_r(\mathbf{0}).$$

Similarly,

$$D_r f_\theta(\mathbf{0}) = \lim_{(x,y)\to\mathbf{0}} \frac{f_\theta(x,y)}{|(x,y)|} = \lim_{(x,y)\to\mathbf{0}} g_\theta(x,y) = D_1F_2(\mathbf{0}) = -D_2F_1(\mathbf{0}),$$

so that

$$(\operatorname{curl} F)(\mathbf{0}) = D_1F_2(\mathbf{0}) - D_2F_1(\mathbf{0}) = 2D_r f_\theta(\mathbf{0}).$$

□

If F is a velocity field then the limit in the formula

$$(\operatorname{curl} F)(\mathbf{0}) = 2 \lim_{r\to 0^+} \frac{f_\theta(r\cos\theta, r\sin\theta)}{r}$$

has the interpretation of the angular velocity of F at the origin. That is, when the Cauchy–Riemann equations hold,

the curl is twice the angular velocity.

Indeed, the angular velocity ω away from the origin is by definition the rate of increase of the polar angle θ with the motion of F. This is not the counterclockwise component f_θ, but rather $\omega = f_\theta/r$, i.e., ω is the function called g_θ in the proof of Proposition 1. To understand this, think of a uniformly spinning disk such as a record on a turntable. At each point except the center, the angular velocity is the same. But the speed of motion is not constant over the disk, it is the angular velocity times the distance from the center. That is, the angular velocity is the speed divided by the radius, as claimed. In these terms, the proof showed that the angular velocity ω extends continuously to $\mathbf{0}$, and that $(\operatorname{curl} F)(\mathbf{0})$ is twice the extended value $\omega(\mathbf{0})$.

Also, if F is a velocity field then the right side of the formula

$$(\operatorname{div} F)(\mathbf{0}) = 2 \lim_{r\to 0^+} \frac{f_r(r\cos\theta, r\sin\theta)}{r}$$

has the interpretation of the flux density of F at the origin. That is, when the Cauchy–Riemann equations hold,

the divergence is the flux density.

To understand this, think of a planar region of incompressible fluid about the origin, and let r be a positive number small enough that the fluid fills the area inside the circle of radius r. Suppose that new fluid being added throughout the interior of the circle, at rate c per unit of area. Thus fluid is being added to the area inside the circle at total rate $\pi r^2 c$. Here c is called the flux density over the circle and it is is measured in reciprocal time units, while the units of $\pi r^2 c$ are area over time. Since the fluid is incompressible, $\pi r^2 c$ is also the rate at which fluid is passing normally outward through the circle. And since the circle has circumference $2\pi r$, fluid is therefore passing normally outward through each point of the circle with radial velocity

$$f_r(r\cos\theta, r\sin\theta) = \frac{\pi r^2 c}{2\pi r} = \frac{rc}{2}.$$

Consequently,

$$2\frac{f_r(r\cos\theta, r\sin\theta)}{r} = c.$$

Now let r shrink to 0. The left side of the display goes to the divergence of F at $\mathbf{0}$, and the right side becomes the continuous extension to radius 0 of the flux density over the circle of radius r. That is, the divergence is the flux density when fluid is being added at a single point.