FORMS OF THE CAUCHY–RIEMANN EQUATIONS

Let Ω be a region in $\mathbf{C}$, and let $f : \Omega \longrightarrow \mathbf{C}$ be a function. View points of the domain Ω either as vectors $z = (x, y)$ or as complex numbers $z = x + iy$. View f either as a vector-valued function $f = (u, v)$ or as a complex-valued function $f = u + iv$. The Cauchy–Riemann equations take the following four forms:

- Input and output in coordinates, viewing f as complex-valued ($z = x + iy$, $f(z) = u(x, y) + iv(x, y)$):

  $$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

  in which case the derivative of f has four equivalent forms,

  $$f' = u_x + iv_x = u_x - iu_y = v_y + iv_x = v_y - iu_y.$$

- Input and output in coordinates, viewing f as vector-valued ($z = (x, y)$, $f(z) = (u(x, y), v(x, y))$):

  $$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

  in which case the derivative matrix of f is skew symmetric,

  $$f' = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{where } a = u_x \text{ and } b = v_x.$$

- Input in coordinates, output in complex form ($z = x + iy$, $f(z)$):

  $$f_x = -if_y,$$

  in which case the derivative of f has two equivalent forms,

  $$f' = f_x = -if_y.$$

- Input and output both in complex form (z, $f(z)$): Define differential operators

  $$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

  Then f is annihilated by the second of these,

  $$\frac{\partial f}{\partial \overline{z}} = 0,$$

  and its derivative is given by the first,

  $$f' = \frac{\partial f}{\partial z}.$$

One should apply whichever version of the Cauchy–Riemann equations best fit a given context.

The differential operators in the fourth version of the Cauchy–Riemann equations are suggested by the relations

$$x = \frac{1}{2}(z + \overline{z}), \qquad y = -\frac{i}{2}(z - \overline{z}),$$

and by the chain rule. The symbol-patterns are

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial}{\partial x}\frac{1}{2} - \frac{\partial}{\partial y}\frac{i}{2} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

and similarly for $\partial/\partial\overline{z}$. These calculations are not meaningful analytically because z and $\overline{z}$ are not independent real variables, but in practice they tend to work. For example, if

$$f(z) = |z|^2 = z\overline{z},$$

then also $f(z) = x^2 + y^2$, so that $\partial f/\partial\overline{z} = (2x + 2iy)/2 = x + iy$. That is, exactly as one would expect,

$$\frac{\partial f}{\partial\overline{z}} = z.$$

And this calculation shows that that f does not satisfy the Cauchy–Riemann equations except at $z = 0$.

For a discussion of the Cauchy–Riemann equations and polar coordinates, see the related writeup “Geometry of the Cauchy–Riemann equations.”