1. Cauchy’s Theorem for Triangles

Let $\Omega$ be a region, let $f: \Omega \rightarrow \mathbb{C}$ be differentiable, and let $T$ be a triangle in $\Omega$. Here $T$ is not only three line segments, but also the region that they surround. The three segments, traversed counterclockwise, are denoted $T$. That is, $T = \partial T$. Cauchy’s Theorem for triangles states that

$$\int_T f(z) \, dz = 0.$$ 

Bisect each side of $T$ to get four counterclockwise triangles $T^j_1 = \partial T^j_1$ for $j = 1, \ldots, 4$. Then

$$\left| \int_T f(z) \, dz \right| = \sum_{j=1}^{4} \int_{T^j_1} f(z) \, dz \leq \sum_{j=1}^{4} \int_{T^j_1} f(z) \, dz \leq 4 \int_{T_1} f(z) \, dz,$$

where $T_1$ is one of the triangles $T^j_1$. Iterating the argument shows that for any $n > 0$ we have

$$\left| \int_T f(z) \, dz \right| \leq 4^n \left| \int_{T_n} f(z) \, dz \right|,$$

where $T_n = \partial T_n$ and $T \supset T_1 \supset \cdots \supset T_n$, each triangle’s sides being half as long as those of the triangle before it.

The intersection of all the solid triangles is a single point,

$$\bigcap_{n \geq 1} T_n = \{p\}.$$

Indeed, since the triangle diameters shrink by a factor of two at each generation, the intersection can’t be more than one point. And since no finite intersection of $T_n$’s is empty, the infinite intersection isn’t empty either because $T$ is compact. For convenience we may assume that $p = 0$.

The derivative $f'(0)$ exists, meaning that

$$f(z) = f(0) + f'(0)z + o(z).$$

Consequently,

$$\int_{T_n} f(z) \, dz = f(0) \int_{T_n} dz + f'(0) \int_{T_n} z \, dz + \int_{T_n} o(z) \, dz.$$

But 1 and $z$ have antiderivatives, and $T_n$ is closed, so in fact

$$\int_{T_n} f(z) \, dz = \int_{T_n} o(z) \, dz.$$
Let $\varepsilon > 0$ be given. Since the triangles $\{T_n\}$ are shrinking to 0, we have $o(z) \leq \varepsilon |z|$ for all $z \in T_n$ as soon as $n$ is large enough. For such $n$, 

$$\left| \int_{T_n} f(z) \, dz \right| \leq \varepsilon \sup \{ |z| : z \in T_n \} \cdot \text{length}(T_n) \leq \varepsilon \cdot (\text{length}(T_n))^2.$$ 

But $\text{length}(T_n) = \text{length}(T)/2^n$. Therefore, for large enough $n$, 

$$\left| \int_{T_n} f(z) \, dz \right| \leq \varepsilon \cdot (\text{length}(T))^2/4^n.$$ 

Combine inequality (1) with these results to get 

$$\left| \int_T f(z) \, dz \right| \leq 4^n \left| \int_{T_n} f(z) \, dz \right| \leq \varepsilon \cdot (\text{length}(T))^2.$$ 

Since $\varepsilon > 0$ is arbitrary and $\text{length}(T)$ is finite, the desired result follows, 

$$\int_T f(z) \, dz = 0.$$ 

2. **Cauchy’s Theorem for Simple Polygons**

Let $\Omega$ be a region, let $f : \Omega \rightarrow \mathbb{C}$ be differentiable, and let $P$ be a simple polygon in $\Omega$. Here $P$ is not only the boundary segments, but also the region that they surround. The segments, traversed counterclockwise, are denoted $P$. That is, $P = \partial P$. To say that the polygon is simple is to say that the only intersection points of the segments are each segment’s endpoint and the start-point of the next segment, and the last segment’s endpoint and the start-point of the first segment. Cauchy’s Theorem for simple polygons states that 

$$\int_P f(z) \, dz = 0.$$ 

The idea is that if the polygon is convex then it can be dissected into finitely many triangles, making the result immediate. And if the polygon is not convex then a combinatorial argument shows that it can be dissected into finitely many convex polygons.

3. **Cauchy’s Theorem for Simple Curves**

Let $\Omega$ be a region, let $f : \Omega \rightarrow \mathbb{C}$ be differentiable, and let $\gamma$ be a simple rectifiable closed curve in $\Omega$ whose interior lies in $\Omega$. (A simple closed curve is a loop with no self-intersections except that its endpoint is its start-point. A rectifiable curve is a curve of finite length. The seemingly self-evident fact that a simple closed curve has an interior and an exterior is the Jordan Curve Theorem, not at all trivial to prove. For example, the theorem fails for simple closed curves on a torus rather than in the plane, even though the plane and the torus are indistinguishable in the small; so the proof must make use of something quantifiable that distinguishes the plane from the torus.) Cauchy’s Theorem for simple curves states that 

$$\int_\gamma f(z) \, dz = 0.$$
The proof requires a little topology. The first claim is that for some $\rho > 0$, the $\rho$-thickened version of the curve still lies in the region,

$$\bigcup_{z \in \gamma} B(z, \rho) \subset \Omega. \quad (2)$$

(Here $\overline{B}(z, \rho)$ is the closed ball about $z$ of radius $\rho$.) The containment is obvious if $\Omega$ is all of $\mathbb{C}$. Otherwise, the complement $\Omega^c$ is nonempty, and so we can define the distance function from the curve to the complement

$$d : \gamma \rightarrow \mathbb{R}^+, \quad d(z) = \inf\{|z - w| : w \in \Omega^c\}.$$ 

The function is continuous, and $\gamma$ is compact, and so the function takes a minimum, which is positive. Let this minimum be $2\rho$. Then

$$|z - w| \geq 2\rho \quad \text{for all } z \in \gamma \text{ and } w \in \Omega^c,$$

and the desired containment (2) follows.

Let $R$ (for “ribbon”) denote the thickened curve,

$$R = \bigcup_{z \in \gamma} \overline{B}(z, \rho) \subset \Omega.$$

Consider finitely many points $z_0, z_1, z_2, \ldots, z_n = z_0$ of $\gamma$, in order of clockwise traversal, each within distance $\rho$ of the next along the length of $\gamma$. Consecutive points must then also be within distance $\rho$ of each other in $\mathbb{C}$, and so the polygon $P$ with the points as vertices lies in the ribbon $R$. If we add more points, this will not increase the distances between consecutive points along the curve, and so the resulting new polygon will still lie in $R$. (But note that we needed to be a bit careful here: the weaker property that consecutive points along the curve be within distance $\rho$ of each other in $\mathbb{C}$ need not be preserved under the addition of more points: two points leading in and out of a hairpin turn are close, but the point at the turn is far from them both.)

Consider the sum

$$S = \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1}).$$

This is a Riemann sum for the curve integral $\int_{\gamma} f(z) \, dz$ that we want to equal zero, and by taking enough division points $z_j$ we can make $S$ as close to $\int_{\gamma} f(z) \, dz$ as we wish.

Also, $S$ is a Riemann sum for the polygon integral $\int_P f(z) \, dz$ that we already know equals zero. However, the argument that adding more division points thus also makes $S$ as close to zero as we wish isn’t quite transparent. The problem is that while the curve $\gamma$ is fixed in this discussion, so that adding points in the previous paragraph refined Riemann sums for the one particular integral $\int_{\gamma} f(z) \, dz$, adding points also changes the polygon $P$ and hence changes the integral $\int_P f(z) \, dz$ being approximated by the Riemann sum $S$. So even though polygon integrals are zero, a little more work is required to show that adding enough points makes $S$ close to zero by making it close to a polygon integral.

The difference between the polygon integral and the sum is

$$\int_P f(z) \, dz - S = \sum_{j=1}^{n} \left( \int_{z_{j-1}}^{z_j} f(z) \, dz - f(z_j)(z_j - z_{j-1}) \right),$$

but...
where the integrals are along the line segments $[z_{j-1}, z_j]$ joining the endpoints. The equality rewrites as

$$\int_P f(z) \, dz - S = \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} (f(z) - f(z_j)) \, dz.$$ 

As explained above, the polygon $P$ remains in the ribbon $R$ as we add points. Also, the polygon $P$ remains an inscribed polygon of $\gamma$ as points are added, so that always

$$\text{length}(P) \leq \text{length}(\gamma).$$

Since $R$ is a compact subset of $\Omega$, and since $f$ is continuous on $\Omega$, $f$ is uniformly continuous on $R$. So, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $z, \tilde{z} \in R$,

$$|z - \tilde{z}| < \delta \implies |f(z) - f(\tilde{z})| < \varepsilon/\text{length}(\gamma).$$

Finally, add enough points to make $|z_j - z_{j-1}| < \delta$ for all $j$. This puts everything in place for the final calculation,

$$\left| \int_P f(z) \, dz - S \right| = \left| \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} (f(z) - f(z_j)) \, dz \right|$$

$$\leq \sum_{j=1}^{n} \left| \int_{z_{j-1}}^{z_j} (f(z) - f(z_j)) \, dz \right|$$

$$\leq \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} |f(z) - f(z_j)| \, |dz|$$

$$\leq \sum_{j=1}^{n} \sup_{z \in [z_{j-1}, z_j]} \{|f(z) - f(z_j)| \cdot |z_j - z_{j-1}|}$$

$$\leq \sum_{j=1}^{n} (\varepsilon/\text{length}(\gamma)) \cdot |z_j - z_{j-1}|$$

$$\leq \varepsilon \cdot \text{length}(P)$$

$$\leq \varepsilon.$$

Since the sum $S$ is arbitrarily close to the curve-integral $\int_{\gamma} f(z) \, dz$, and since it is arbitrary close to the polygon integral $\int_P f(z) \, dz = 0$, Cauchy’s Theorem for simple curves follows,

$$\int_{\gamma} f(z) \, dz = 0.$$