CAUCHY'S THEOREM FOR SIMPLE CURVES

1. Cauchy's Theorem for Triangles

Let Ω be a region, let $f : \Omega \longrightarrow \mathbf{C}$ be differentiable, and let $\mathbb{T}$ be a triangle in Ω. Here $\mathbb{T}$ is not only three line segments, but also the region that they surround. The three segments, traversed counterclockwise, are denoted T. That is, $T = \partial\mathbb{T}$. Cauchy's Theorem for triangles states that

$$\int_T f(z)\,dz = 0.$$

Bisect each side of T to get four counterclockwise triangles $T_1^j = \partial\mathbb{T}_1^j$ for $j = 1, \dots, 4$. Then

$$\left|\int_T f(z)\,dz\right| = \left|\sum_{j=1}^{4}\int_{T_1^j} f(z)\,dz\right| \le \sum_{j=1}^{4}\left|\int_{T_1^j} f(z)\,dz\right| \le 4\left|\int_{T_1} f(z)\,dz\right|,$$

where T_1 is one of the triangles T_1^j. Iterating the argument shows that for any $n > 0$ we have

$$\left|\int_T f(z)\,dz\right| \le 4^n\left|\int_{T_n} f(z)\,dz\right|, \tag{1}$$

where $T_n = \partial\mathbb{T}_n$ and $\mathbb{T} \supset \mathbb{T}_1 \supset \cdots \supset \mathbb{T}_n$, each triangle's sides being half as long as those of the triangle before it.

The intersection of all the solid triangles is a single point,

$$\bigcap_{n\ge 1}\mathbb{T}_n = \{p\}.$$

Indeed, since the triangle diameters shrink by a factor of two at each generation, the intersection can't be more than one point. And since no finite intersection of $\mathbb{T}_n$'s is empty, the infinite intersection isn't empty either because $\mathbb{T}$ is compact.

The derivative $f'(p)$ exists, meaning that

$$\lim_{z\to p}\frac{\eta(z)}{z-p} = 0 \quad \text{where } \eta(z) = f(z) - f(p) - f'(p)(z-p).$$

By definition of η,

$$\int_{T_n} f(z)\,dz = \int_{T_n} (f(p) + f'(p)(z-p) + \eta(z))\,dz.$$

But $f(p)$ and $f'(p)(z-p)$ have antiderivatives, and T_n is closed, so that in fact

$$\int_{T_n} f(z)\,dz = \int_{T_n} \eta(z)\,dz,$$

and therefore

$$\left|\int_{T_n} f(z)\,dz\right| \le \sup_{z\in T_n}\{|\eta(z)|\}\cdot \text{length}(T_n).$$

Let $\varepsilon > 0$ be given. For all large enough n we have

$$|\eta(z)| < \varepsilon \cdot |z - p| \quad \text{for all } z \in T_n,$$

so that

$$|\eta(z)| < \varepsilon \cdot \text{length}(T_n) \quad \text{for all } z \in T_n,$$

and consequently,

$$\left| \int_{T_n} f(z)\, dz \right| \leq \varepsilon \cdot (\text{length}(T_n))^2.$$

But the length of T_n is the length of T divided by 2^n. Therefore, for large enough n,

$$\left| \int_{T_n} f(z)\, dz \right| \leq \varepsilon \cdot (\text{length}(T))^2 / 4^n.$$

Combine inequality (1) with these results to get

$$\left| \int_{T} f(z)\, dz \right| \leq 4^n \left| \int_{T_n} f(z)\, dz \right| \leq \varepsilon \cdot (\text{length}(T))^2.$$

Since $\varepsilon > 0$ is arbitrary and the length of T is finite, the desired result follows,

$$\int_T f(z)\, dz = 0.$$

2. Cauchy's Theorem for Simple Polygons

Let Ω be a region, let $f : \Omega \longrightarrow \mathbf{C}$ be differentiable, and let $\mathbb{P}$ be a simple polygon in Ω. Here $\mathbb{P}$ is not only the boundary segments, but also the region that they surround. The segments, traversed counterclockwise, are denoted P. That is, $P = \partial\mathbb{P}$. To say that the polygon is *simple* is to say that the only intersection points of the segments are each segment's endpoint and the start-point of the next segment, and the last segment's endpoint and the start-point of the first segment. Cauchy's Theorem for simple polygons states that

$$\int_P f(z)\, dz = 0.$$

The idea is that if the polygon is convex then it can be dissected into finitely many triangles, making the result immediate. And if the polygon is not convex then a combinatorial argument shows that it can be dissected into finitely many convex polygons.

3. Cauchy's Theorem for Simple Curves

Let Ω be a region, let $f : \Omega \longrightarrow \mathbf{C}$ be differentiable, and let γ be a simple rectifiable closed curve in Ω whose interior lies in Ω. (A *simple closed* curve is a loop with no self-intersections except that its endpoint is its start-point. A *rectifiable* curve is a curve of finite length. The seemingly self-evident fact that a simple closed curve has an interior and an exterior is the *Jordan Curve Theorem*, not at all trivial to prove. For example, the theorem fails for simple closed curves on a torus rather than in the plane, even though the plane and the torus are indistinguishable in the small; so the proof must make use of something quantifiable that distinguishes the plane from the torus.) Cauchy's Theorem for simple curves states that

$$\int_\gamma f(z)\, dz = 0.$$

The proof requires a little topology. The first claim is that for some $\rho > 0$, the ρ-thickened version of the curve still lies in the region,

$$\bigcup_{z\in\gamma} \overline{B(z,\rho)} \subset \Omega. \tag{2}$$

(Here $\overline{B(z,\rho)}$ is the *closed* ball about z of radius ρ.) The containment is obvious if Ω is all of $\mathbf{C}$. Otherwise, the complement Ω^c is nonempty, and so we can define the distance function from the curve to the complement

$$d : \gamma \longrightarrow \mathbf{R}^+, \qquad d(z) = \inf\{|z - w| : w \in \Omega^c\}.$$

The function is continuous, and γ is compact, and so the function takes a minimum, which is positive. Let this minimum be 2ρ. Then

$$|z - w| \geq 2\rho \quad \text{for all } z \in \gamma \text{ and } w \in \Omega^c,$$

and the desired containment (2) follows.

Let R (for "ribbon") denote the thickened curve,

$$R = \bigcup_{z\in\gamma} \overline{B(z,\rho)} \subset \Omega.$$

Consider finitely many points z_0, z_1, z_2, $\dots$, $z_n = z_0$ of γ, in order of clockwise traversal, each within distance ρ of the next along the length of γ. Consecutive points must then also be within distance ρ of each other in $\mathbf{C}$, and so the polygon P with the points as vertices lies in the ribbon R. If we add more points, this will not increase the distances between consecutive points along the curve, and so the resulting new polygon will still lie in R. (But note that we needed to be a bit careful here: the weaker property that consecutive points along the curve be within distance ρ of each other in $\mathbf{C}$ need *not* be preserved under the addition of more points: two points leading in and out of a hairpin turn are close, but the point at the turn is far from them both.)

Consider the sum

$$S = \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1}).$$

This is a Riemann sum for the curve integral $\int_\gamma f(z)\,dz$ that we want to equal zero, and by taking enough division points z_j we can make S as close to $\int_\gamma f(z)\,dz$ as we wish.

Also, S is a Riemann sum for the polygon integral $\int_P f(z)\,dz$ that we already know equals zero. However, the argument that adding more division points thus also makes S as close to zero as we wish isn't quite transparent. The problem is that while the curve γ is fixed in this discussion, so that adding points in the previous paragraph refined Riemann sums for the *one particular* integral $\int_\gamma f(z)\,dz$, adding points also changes the polygon P and hence changes the integral $\int_P f(z)\,dz$ being approximated by the Riemann sum S. So even though polygon integrals are zero, a little more work is required to show that adding enough points makes S close to zero by making it close to a polygon integral.

The difference between the polygon integral and the sum is

$$\int_P f(z)\,dz - S = \sum_{j=1}^{n} \left(\int_{z_{j-1}}^{z_j} f(z)\,dz - f(z_j)(z_j - z_{j-1}) \right),$$

where the integrals are along the line segments $[z_{j-1}, z_j]$ joining the endpoints. The equality rewrites as

$$\int_P f(z)\,dz - S = \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} (f(z) - f(z_j))\,dz.$$

As explained above, the polygon P remains in the ribbon R as we add points. Also, the polygon P remains an inscribed polygon of γ as points are added, so that always

$$\text{length}(P) \leq \text{length}(\gamma).$$

Since R is a compact subset of Ω, and since f is continuous on Ω, f is uniformly continuous on R. So, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $z, \tilde{z} \in R$,

$$|z - \tilde{z}| < \delta \implies |f(z) - f(\tilde{z})| < \varepsilon/\text{length}(\gamma).$$

Finally, add enough points to make $|z_j - z_{j-1}| < \delta$ for all j. This puts everything in place for the final calculation,

$$\begin{aligned}
\left| \int_P f(z)\,dz - S \right| &= \left| \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} (f(z) - f(z_j))\,dz \right| \\
&\leq \sum_{j=1}^{n} \left| \int_{z_{j-1}}^{z_j} (f(z) - f(z_j))\,dz \right| \\
&\leq \sum_{j=1}^{n} \int_{z_{j-1}}^{z_j} |f(z) - f(z_j)|\,|dz| \\
&\leq \sum_{j=1}^{n} \sup_{z \in [z_{j-1}, z_j]} \{|f(z) - f(z_j)|\} \cdot |z_j - z_{j-1}| \\
&\leq \sum_{j=1}^{n} (\varepsilon/\text{length}(\gamma)) \cdot |z_j - z_{j-1}| \\
&\leq (\varepsilon/\text{length}(\gamma)) \cdot \text{length}(P) \\
&\leq \varepsilon.
\end{aligned}$$

Since the sum S is arbitrarily close to the curve-integral $\int_\gamma f(z)\,dz$, and since it is arbitrary close to the polygon integral $\int_P f(z)\,dz = 0$, Cauchy's Theorem for simple curves follows,

$$\int_\gamma f(z)\,dz = 0.$$