THE BERNOULLI NUMBERS AND POWER SUMS

The Bernoulli numbers, which we encountered in the homework, arise naturally in the context of computing the *power sums*

$$\begin{aligned} 1^0 + 2^0 + \cdots + n^0 &= n, \\ 1^1 + 2^1 + \cdots + n^1 &= \frac{1}{2}(n^2 + n), \\ 1^2 + 2^2 + \cdots + n^2 &= \frac{1}{6}(2n^3 + 3n^2 + n), \\ \text{etc.} \end{aligned}$$

To study these, let n be a positive integer and let the kth power sum up to $n-1$ be

$$S_k(n) = \sum_{m=0}^{n-1} m^k, \quad k \in \mathbb{N}.$$

Thus $S_0(n) = n$ while for $k > 0$ the term 0^k is 0. (Having the sum start at 0 and stop at $n-1$ neatens the ensuing calculation.) The power series having these sums as its coefficients is their *generating function*,

$$\mathbf{S}(n,t) = \sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!}.$$

A little work shows that rearranging the double sum gives

$$\mathbf{S}(n,t) = \frac{e^{nt} - 1}{t} \, \frac{t}{e^t - 1}.$$

The second term is independent of n. As in the homework, its coefficients are by definition the *Bernoulli numbers*, constants that can be computed once and for all,

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Now a little more work shows that the generating function rearranges to

$$\begin{aligned} \mathbf{S}(n,t) &= \sum_{\ell=1}^{\infty} n^\ell \frac{t^{\ell-1}}{\ell!} \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} \right) \frac{t^k}{k!}. \end{aligned}$$

Thus, if we define the kth *Bernoulli polynomial* as

$$B_k(X) = \sum_{j=0}^{k} \binom{k}{j} B_j X^{k-j},$$

which again can be computed once and for all, then comparing the first and last expressions for $\mathbf{S}(n,t)$ shows that the kth power sum is

$$S_k(n) = \frac{1}{k+1}(B_{k+1}(n) - B_{k+1}).$$

From the homework, the first few Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_3 = 0$, and so the first few Bernoulli polynomials are

$$\begin{aligned} B_0(X) &= 1, \\ B_1(X) &= X - \frac{1}{2}, \\ B_2(X) &= X^2 - X + \frac{1}{6}, \\ B_3(X) &= X^3 - \frac{3}{2}X^2 + \frac{1}{2}X. \end{aligned}$$

For example,

$$1^2 + 2^2 + \cdots + n^2 = S_2(n+1) = \frac{B_3(n+1) - B_3}{3}$$

works out to $(2n^3 + 3n^2 + n)/6$ as it should.