THE ARZELA–ASCOLI THEOREM

Let $\Omega$ be a region in $\mathbb{C}$. Let $\Omega_Q$ denote its subset of points with rational coordinates,

$$\Omega_Q = \{x + iy \in \Omega : x, y \in \mathbb{Q}\}.$$

This subset is useful because it is small in the sense that is countable, but large in the sense that it is dense in $\Omega$.

**Definition 0.1.** A family $\mathcal{F}$ of complex-valued functions on $\Omega$ is pointwise bounded if

$$\text{for each } z \in \Omega, \quad \sup_{f \in \mathcal{F}} \{|f(z)|\} < \infty.$$ 

This does not imply that any $f \in \mathcal{F}$ is bounded on $\Omega$, as demonstrated by the family $\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\}$ where $f_n(z) = z/n$ for $z \in \mathbb{C}$.

Nor is it implied if every $f \in \mathcal{F}$ is bounded on $\Omega$, as demonstrated by the family $\mathcal{F} = \{f_n : n \in \mathbb{Z}^+\}$ where $f_n(z) = n$ for $z \in \mathbb{C}$.

**Definition 0.2.** A family $\mathcal{F}$ of complex-valued functions on $\Omega$ is equicontinuous if for every $\varepsilon > 0$ and $z \in \Omega$, there exists some $\delta > 0$ such that for all $\tilde{z} \in \Omega$,

$$|\tilde{z} - z| < \delta \implies |f(\tilde{z}) - f(z)| < \varepsilon \text{ for all } f \in \mathcal{F}.$$ 

The idea here is that each $f \in \mathcal{F}$ is pointwise continuous on $\Omega$, and at each point $z \in \Omega$, given $\varepsilon > 0$, the same $\delta > 0$ works simultaneously for all $f \in \mathcal{F}$ in the definition of continuity at $z$.

**Theorem 0.3** (Arzela–Ascoli). Let $\Omega$ be a region in $\mathbb{C}$, and let $\mathcal{F}$ be a pointwise bounded, equicontinuous family of complex-valued functions on $\Omega$. Then every sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence that converges to a continuous function on $\Omega$, the convergence being uniform on compact subsets.

**Proof.** Let $\{f_n\}$ be a sequence in $\mathcal{F}$.

First we use the given pointwise boundedness to prove that $\{f_n\}$ has a subsequence that converges on $\Omega_Q$. The idea is a variant of Cantor’s diagonal argument. Since $\Omega_Q$ is countable, write

$$\Omega_Q = \{z_1, z_2, z_3, \ldots\}.$$

The complex sequence

$$\{f_1(z_1), f_2(z_1), f_3(z_1), \ldots\}$$

is bounded, and so it contains a convergent subsequence. Relabel the convergent subsequence as follows:

$$\{f_{1,1}(z_1), f_{1,2}(z_1), f_{1,3}(z_1), \ldots\} \text{ converges.}$$

Next, the complex sequence

$$\{f_{1,1}(z_2), f_{1,2}(z_2), f_{1,3}(z_2), \ldots\}$$
is again bounded, so it too contains a convergent subsequence. Relabel it:
\[ \{f_{2,1}(z_2), f_{2,2}(z_2), f_{2,3}(z_2), \ldots \} \] converges.

And the complex sequence
\[ \{f_{2,1}(z_3), f_{2,2}(z_3), f_{2,3}(z_3), \ldots \} \]
is bounded, so it contains a convergent subsequence:
\[ \{f_{3,1}(z_3), f_{3,2}(z_3), f_{3,3}(z_3), \ldots \} \] converges.

Continuing this process gives rise to an array,
\[
\begin{array}{cccc}
  f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\
  f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\
  f_{3,1} & f_{3,2} & f_{3,3} & \cdots \\
  \vdots & \vdots & \vdots & \ddots
\end{array}
\]
The first row is a sequence of functions that converges at \( z_1 \). The second row is a subsequence of the first row, and it converges at \( z_1 \) and at \( z_2 \). The third row is a subsequence of the second row, and it converges at \( z_1 \), at \( z_2 \), and at \( z_3 \). And so on. Consider the sequence down the diagonal,
\[ \{f_{1,1}, f_{2,2}, f_{3,3}, \ldots \}. \]

This is a subsequence of the original sequence \( \{f_n\} \), and it converges at each \( z \in \Omega_Q \).

After relabeling, we may assume that the original sequence \( \{f_n\} \) converges on \( \Omega_Q \).

Next we use the given equicontinuity to prove that in fact \( \{f_n\} \) converges on all of \( \Omega \). This is a typical three-epsilon argument. Let \( \varepsilon > 0 \) be given, and let \( z \in \Omega \) be given. The equicontinuity of \( F \) supplies a corresponding \( \delta = \delta(\varepsilon, z) > 0 \) such that for \( \tilde{z} \in \Omega \),
\[ |\tilde{z} - z| < \delta \implies |f(\tilde{z}) - f(z)| < \varepsilon \text{ for all } f \in F. \]

Because \( \Omega_Q \) is dense in \( \Omega \), there exists a point \( z_Q \in \Omega_Q \) such that
\[ |z_Q - z| < \delta. \]

Because the complex sequence \( \{f_n(z_Q)\} \) converges, it is Cauchy, meaning that there exists a starting index \( N = N(\varepsilon, z_Q) \) such that for all integers \( n \) and \( m \),
\[ n, m > N \implies |f_n(z_Q) - f_m(z_Q)| < \varepsilon. \]

Consequently, the complex sequence \( \{f_n(z)\} \) is Cauchy as well,
\[ n, m > N \implies |f_n(z) - f_m(z)| < 3\varepsilon. \]

Since the complex sequence \( \{f_n(z)\} \) is Cauchy, it converges.

Third we prove that the pointwise limit function
\[ g = \lim_n f_n : \Omega \to \mathbb{C} \]
is continuous. Let \( \varepsilon > 0 \) be given, and let \( z \in \Omega \) be given. The equicontinuity of \( F \) supplies a corresponding \( \delta = \delta(\varepsilon, z) > 0 \) such that for \( \tilde{z} \in \Omega \),
\[ |\tilde{z} - z| < \delta \implies |f(\tilde{z}) - f(z)| < \varepsilon \text{ for all } f \in F. \]
Consider any $\tilde{z} \in \Omega$ such that $|\tilde{z} - z| < \delta$. By the pointwise convergence of $\{f_n\}$ to $g$, for some starting index $N = N(\epsilon, z, \tilde{z})$ we have
\[
\begin{align*}
|f_n(z) - g(z)| < \epsilon \\
|f_n(\tilde{z}) - g(\tilde{z})| < \epsilon
\end{align*}
\]
for all $n > N$.

Thus
\[
|g(\tilde{z}) - g(z)| \leq \left( \begin{align*}
|g(\tilde{z}) - f_n(\tilde{z})| + |f_n(\tilde{z}) - f_n(z)| + |f_n(z) - g(z)|
\end{align*} \right) < 3\epsilon
\]
for all $n > N$.

But $g(\tilde{z}) - g(z)$ is independent of $n$, so the “for all $n > N$” in the display is irrelevant: for all $\tilde{z} \in \Omega$ such that $|\tilde{z} - z| < \delta$,
\[
|g(\tilde{z}) - g(z)| < 3\epsilon.
\]
That is, $g$ is continuous at $z$, as desired.

Finally we prove that the convergence of $\{f_n\}$ to $g$ is uniform on compact subsets of $\Omega$. Let $K$ be such a compact set, and let $\epsilon > 0$ be given. We seek a starting index $N = N(\epsilon, K)$ such that for all integers $n$,
\[
N > N \implies |f_n(z) - g(z)| < 3\epsilon \quad \text{for all } z \in K.
\]

Let $z \in K$ be given. The equicontinuity of $F \cup \{g\}$ supplies a corresponding $\delta = \delta_z = \delta(\epsilon, z) > 0$ such that for $\tilde{z} \in K$,
\[
|\tilde{z} - z| < \delta_z \implies \left\{ \begin{align*}
|f_n(\tilde{z}) - f_n(z)| < \epsilon & \quad \text{for all } n \in \mathbb{Z}^+ \\
|g(\tilde{z}) - g(z)| < \epsilon
\end{align*} \right. \}
\]
and because $\{f_n(z)\}$ converges to $g(z)$, there exists some $N = N_z = N(\epsilon, z) \in \mathbb{Z}^+$ such that for all integers $n$,
\[
N > N_z \implies |f_n(z) - g(z)| < \epsilon.
\]

So, for all $\tilde{z} \in K$ and all integers $n$,
\[
\left\{ \begin{align*}
|\tilde{z} - z| < \delta_z & \quad \text{for all } n > N_z
\end{align*} \right. \} \implies |f_n(\tilde{z}) - g(\tilde{z})| \leq \left( \begin{align*}
|f_n(\tilde{z}) - f_n(z)| + |f_n(z) - g(z)| + |g(z) - g(\tilde{z})|
\end{align*} \right) < 3\epsilon.
\]

This shows that the sequence $\{f_n\}$ converges uniformly on $B(z, \delta_z) \cap K$. So consider an open cover of the compact set $K$, $K = \bigcup_{z \in K} B(z, \delta_z) \cap K$.

By compactness, there exists a finite subcover,
\[
K = \bigcup_{j=1}^k B(z_j, \delta_{z_j}) \cap K.
\]
Define
\[
N = \max\{N_{z_1}, \ldots, N_{z_k}\}.
\]
Then for all integers $n$ and $m$, the desired condition holds,

$$n > N \implies |f_n(z) - g(z)| < 3\varepsilon \text{ for all } z \in K.$$ 

This completes the proof. $\square$